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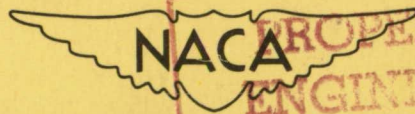
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2004

AN INVESTIGATION OF THE STABILITY OF A SYSTEM COMPOSED
OF A SUBSONIC CANARD AIRFRAME AND A CANTED-AXIS
GYROSCOPE AUTOMATIC PILOT

By Robert A. Gardiner, Jacob Zarovsky,
and H. O. Ankenbruck

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Langley Air Force Base, Va.



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SUMMARY

An analysis has been made of the stability of a system composed of a subsonic canard airframe and a canted-axis gyroscope automatic pilot. The analysis was based on calculations of the airframe frequency response and measurements of the autopilot frequency response. These responses were combined in accordance with the principles of servomechanism theory so that the stability could be determined on the basis of the Nyquist criterion.

INTRODUCTION

The Langley Aeronautical Laboratory of the National Advisory Committee for Aeronautics has conducted an investigation of the stability of a system composed of a subsonic canard airframe and canted-axis gyroscope control equipment. The object of the investigation was to develop a method of analysis that could be used to determine the stability and controllability of the system and to show the application of this method by means of an illustrative example.

The analysis was based on measured autopilot frequency responses obtained from bench tests of the control equipment and calculations of the airframe response. These responses were combined in accordance with the principles of servomechanism theory so that the system stability could be determined on the basis of the Nyquist criterion. The transient response of the airframe and autopilot system to a command signal was determined by the use of a Fourier series technique.

DEFINITIONS AND SYMBOLS

Vector frequency response: A vector which is a frequency-dependent function. The angle of the vector is defined by the phase relationship between the input or forcing function and the output or response function and the magnitude is the ratio of the peak amplitude of the output function to the peak amplitude of the input function when the time variation of the input is sinusoidal.

Closed-loop response: The vector frequency response of a system which is sensitive to error signal. The error is the difference between the input signal and the output signal.

The closed-loop response is equal to the ratio of the output signal to the input signal.

Open-loop response: The vector frequency response of a closed-loop system with the feedback link open. The open link may be indicated by a broken line in block diagrams.

The open-loop response is equal to the ratio of the output signal to the error signal.

Nyquist diagram: A polar plot of the locus of the tip of the open-loop frequency-response vector. This diagram indicates stability of the closed-loop system if the critical point ($-1, 0^\circ$ is the critical point for all diagrams presented herein) is not enclosed by the locus.

Gain margin: On a Nyquist diagram, the percentage gain as measured by the distance from the critical point to the point where the locus cuts the 180° line.

Phase margin: The angle in degrees between the 180° line and the line from the origin to the point where the locus cuts the unit circle on a Nyquist diagram.

Gyroscope cant angle: Angle between the yaw axis and the spin axis when the spin axis lies in the plane of symmetry of the airframe.

Follow-up ratio: The ratio of gyroscope-error angle to control-surface deflection.

Z stability axis which lies in the plane of symmetry and perpendicular to the relative wind and passes through the center of gravity

X stability axis which lies in the plane of symmetry and perpendicular to the Z-axis and passes through the center of gravity

Y	stability axis which passes through the center of gravity and is perpendicular to the plane of symmetry
I_X	moment of inertia about the X-axis, slug-feet square
I_Y	moment of inertia about the Y-axis, slug-feet square
I_Z	moment of inertia about the Z-axis, slug-feet square
k_X	radius of gyration about the X-axis, feet
k_Z	radius of gyration about the Z-axis, feet
m	mass, slugs
S	wing area, square feet
c	wing chord, feet
b	wing span, feet
V	velocity, feet per second
p	rolling angular velocity, degrees per second
q	dynamic pressure, pounds per square foot $\left(\frac{1}{2}\rho V^2\right)$ or pitching angular velocity, degrees per second
r	yawing angular velocity, degrees per second
β	angle of sideslip, positive when the relative wind is to the right of the longitudinal body axis, degrees
α	angle of attack, degrees
η	angle between the longitudinal flight path and the longitudinal principal axis of inertia, positive when principal axis is above the flight path, degrees (fig. 1)
γ	climb or glide angle measured from horizontal, positive in climb, degrees (fig. 1)
e	angle between longitudinal principal axis and body axis, degrees (fig. 1)
θ	pitch angle, nose-up is positive, degrees (fig. 1)

ϕ	roll angle, positive in roll to right, degrees
ψ	yaw angle, positive when nose is to right, degrees
δ_e	elevator angle, positive when trailing edge is down, degrees
δ_a	aileron angle, positive when right aileron is down, degrees
C_L	lift coefficient $\left(\frac{\text{Lift}}{qS}\right)$
C_{L_α}	variation of lift coefficient with angle of attack $\left(\frac{\partial C_L}{\partial \alpha}\right)$
$C_{L_{\delta_e}}$	variation of lift coefficient with elevator angle $\left(\frac{\partial C_L}{\partial \delta_e}\right)$
C_{m_α}	variation of pitching-moment coefficient with angle of attack $\left(\frac{\partial C_m}{\partial \alpha}\right)$
$C_{m_{\delta_e}}$	variation of pitching-moment coefficient with elevator angle $\left(\frac{\partial C_m}{\partial \delta_e}\right)$
C_{m_q}	variation of pitching-moment coefficient with pitching-angular-velocity factor $\left(\frac{\partial C_m}{\partial \frac{qc}{2V}}\right)$
$C_{l_{\delta_a}}$	variation of rolling-moment coefficient with aileron angle $\left(\frac{\partial C_l}{\partial \delta_a}\right)$
C_{l_β}	variation of rolling-moment coefficient with angle of sideslip $\left(\frac{\partial C_l}{\partial \beta}\right)$

$C_{Y\beta}$	variation of side-force coefficient with angle of sideslip $\left(\frac{\partial C_Y}{\partial \beta}\right)$
$C_{n\beta}$	variation of yawing-moment coefficient with angle of sideslip $\left(\frac{\partial C_n}{\partial \beta}\right)$
C_{np}	variation of yawing-moment coefficient with rolling-angular-velocity factor $\left(\frac{\partial C_n}{\partial \frac{pb}{2V}}\right)$
C_{lp}	variation of rolling-moment coefficient with rolling-angular-velocity factor $\left(\frac{\partial C_l}{\partial \frac{pb}{2V}}\right)$
C_{nr}	variation of yawing-moment coefficient with yawing-angular-velocity factor $\left(\frac{\partial C_n}{\partial \frac{rb}{2V}}\right)$
$C_{n\delta_a}$	variation of yawing-moment coefficient with aileron angle $\left(\frac{\partial C_n}{\partial \delta_a}\right)$
$C_{Y\delta_a}$	variation of side-force coefficient with aileron angle $\left(\frac{\partial C_Y}{\partial \delta_a}\right)$
C_{lr}	variation of rolling-moment coefficient with yawing-angular-velocity factor $\left(\frac{\partial C_l}{\partial \frac{rb}{2V}}\right)$
D	differential operator $\left(\frac{d}{dt}\right)$

ω	frequency, radians per second
$i = \sqrt{-1}$	
ϵ	phase angle, positive when output leads input and negative when output lags input, degrees; or error signal
DB	decibel gain where one decibel equals 20 times the logarithm to the base 10 of the absolute value of the ratio of the output function to the input function
A_p	autopilot vector frequency response (δ_e/θ , δ_a/ϕ , or δ_a/ψ)
A_θ , A_ϕ , A_ψ	airframe vector frequency response, mode of motion indicated by subscript

Subscripts:

i	input or forcing function, corresponding to a command calling for a change in heading or to a sinusoidal input variation
o	output or response function, for example, the system response to a command signal or to a sinusoidal input variation

APPARATUS AND TECHNIQUE

Airframe.— The airframe is a canard type. Thrust is derived from a pulse-jet-type motor which is mounted above the fuselage. A photograph of the airframe is included as figure 2. Longitudinal control is attained by the use of horizontal hinged elevator surfaces placed near the nose of the fuselage. Lateral control (both roll and yaw) is obtained from conventional ailerons at the wing trailing edge. Directional stability is obtained from fixed vertical fins at the rear of the fuselage. The airframe has no movable control surface corresponding to a conventional rudder.

Automatic pilot.— The automatic pilot is an elementary, inexpensive, canted-axis, free gyroscope with two sets of electrical contact pick-offs, one set (sensitive to pitch angle) transmitting gyroscope intelligence to the elevators and the other set (sensitive to yaw and roll angle) carrying signals to the ailerons. Each set of pick-offs has a dead spot through which the gyroscope contact must move before an error signal may

be transmitted to the servomotor actuating the corresponding control surfaces. When the dead spot is traversed in response to a disturbance, a signal is carried through the contacts to the servomotor, which moves the control surface and also actuates the autopilot pulley-and-cable follow-up system to center the gyroscope pick-off in the dead spot. The autopilot thus provides on-off control of the constant-speed servomotors and, within the limitations imposed by the dead spot and the limitations imposed by the dynamic response of the constant-speed motors, causes correcting control-surface deflections proportional to the angle to which the airframe is disturbed.

Remote control is provided by precession coils in the autopilot, which, when energized due to a remote-control signal, cause the gyroscope to precess and thus change the space orientation of the airframe. A photograph of the autopilot installed in the airframe is shown as figure 3.

Test apparatus.— The ratios of gyroscope-error angle to control-surface-deflection angle (follow-up ratio) were determined by mounting protractors on the autopilot and control surfaces of the airframe.

An autopilot and two servomotors were mounted so that the follow-up ratios duplicated the ratios used in the airframe. This autopilot servomotor system was mounted on a table which was oscillated sinusoidally in a range of varying amplitudes and frequencies. Table motion and servo motion were measured by NACA control-position recorders. These motions, as well as 0.1-second time signals, were recorded by a galvanometer mounted on a standard NACA instrument base. The autopilot test setup is shown in figure 4.

Preliminary test measurements.— Follow-up ratios were determined by measuring the control-surface deflections resulting from measured angles of gyroscope displacement in the complete airframe. In addition, sufficient measurements were made to determine the ratios of servomotor deflections to corresponding control-surface deflections. These ratios were used to convert servo motion (as measured in the frequency-response tests described subsequently) into control-surface motions. Table I summarizes the autopilot characteristics.

Frequency-response tests.— The vector frequency response of the autopilot servomotor control system was determined by subjecting the autopilot to sinusoidal oscillations. The autopilot was oscillated in roll, pitch, and yaw. All the autopilot frequency-response data used in the analysis were obtained from the test setup which incorporated the same pulley and follow-up arrangement as the airframe.

Inasmuch as preliminary studies of the system indicated a large stability margin for most conditions, frequency-response data were read

by an approximate method, thereby the time and labor involved in data reduction were greatly reduced. The accuracy of these data is therefore estimated at ± 10 percent. Although a more exact method was suggested in reference 1, the method used herein was considered sufficient for this investigation.

No attempt was made to simulate the servo load represented by hinge moments while the autopilot frequency-response characteristics were being determined. Tests showed that when torques equal to the torques due to the hinge moments were applied to the servomotor, negligible effects were produced.

Airframe frequency-response calculations.— The airframe vector-frequency-response calculations were made in accordance with the method outlined in reference 1 for pitch, roll, and yaw, under three flight conditions:

- (a) Level flight at 400 miles per hour
- (b) Level flight at 600 miles per hour
- (c) Climb angle of 3° at 250 miles per hour

Constant speed was assumed for each flight condition, thereby the calculations were simplified.

The equations of motion used for these calculations may be found in the appendix. The values of airframe characteristics used in arriving at the solutions may be found in table II.

METHOD OF ANALYSIS

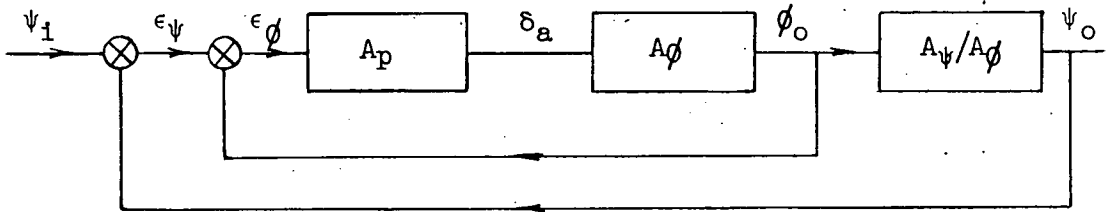
The entire analysis of the system stability is based on the determination of the vector frequency response of the components. Since the airframe is to be stabilized in pitch, roll, and yaw, responses in these three planes are required. Proper manipulation of these responses and proper operation upon them will allow the determination of system stability, of the type of transient response to be expected, and of recommendations for changes in system parameters.

The derivation of open-loop and closed-loop response equations for the system follows:

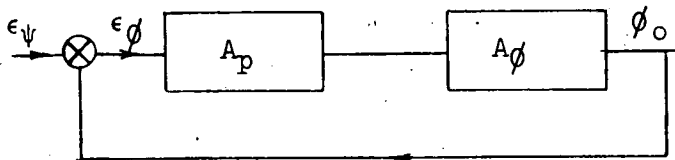
The error angle is denoted by ϵ and is the angle between the gyroscope reference and the instantaneous airframe heading, the

subscript ψ or ϕ indicates an error due to ψ or ϕ . The method of the derivation found herein is carried through in general form in references 2, 3, 4, and 5.

The airframe and autopilot lateral system may be drawn in block diagram form as follows:



This is a multiloop servomechanism and a succession of Nyquist diagrams may be used to determine its stability (reference 2, p. 73). For convenience, the figure may be redrawn to show only the inner loop since its stability must be examined first.



In order to find the open-loop response $\frac{\phi_o}{\epsilon_\phi}$ for this loop the following equations are solved:

$$\delta_a = A_p \epsilon_\phi$$

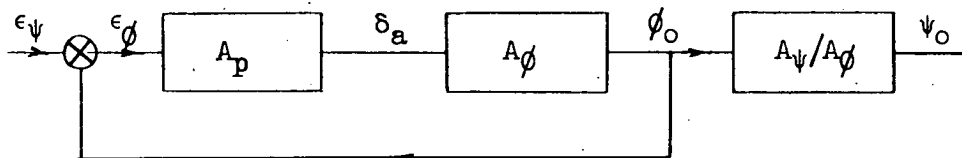
$$\phi_o = A_\phi \delta_a$$

Therefore,

$$\frac{\phi_o}{\epsilon_\phi} = A_p A_\phi$$

From a Nyquist plot of this response, the inner-loop stability may be determined.

If this loop is stable, the usual Nyquist criterion may be applied to the complete lateral system and, for this purpose, the system open-loop response ψ_o/ϵ_ψ is plotted in polar form. This response may be found more easily if the block diagram is redrawn as follows:



The lateral system open-loop response is equal to the inner-loop closed-loop response ϕ_o/ϵ_ψ times the airframe response A_ψ/A_ϕ . The response ϕ_o/ϵ_ψ may be obtained from the following equations:

$$\frac{\phi_o}{\epsilon_\phi} = A_p A_\phi$$

$$\epsilon_\phi = \epsilon_\psi - \phi_o$$

Then

$$\frac{\phi_o}{\epsilon_\psi} = \frac{A_p A_\phi}{1 + A_p A_\phi}$$

The relationship between the open-loop and closed-loop frequency-response vectors is illustrated graphically by figure 5.

The system open-loop response is

$$\frac{\psi_o}{\epsilon_\psi} = \frac{A_p A_\phi}{1 + A_p A_\phi} \frac{A_\psi}{A_\phi}$$

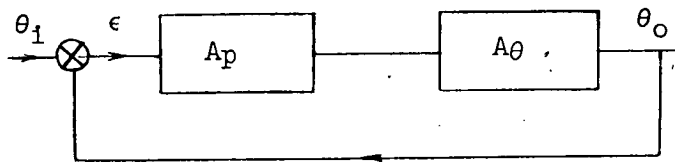
or

$$\frac{\psi_o}{\epsilon_\psi} = \frac{A_p A_\psi}{1 + A_p A_\phi}$$

Since the lateral system closed-loop response may be used to find the yaw transient response, the following derivation is given: For the closed loop, $\epsilon_\psi = \psi_i - \psi_o$. Substituting this value of ϵ_ψ into the expression for the open-loop response gives

$$\frac{\psi_o}{\psi_i} = \frac{A_p A_\psi}{1 + A_p A_\psi + A_p A_\phi}$$

The longitudinal system may be drawn in block diagram form, as follows:



This is a single-loop servomechanism and the open-loop response is θ_o/ϵ , which equals $A_p A_\theta$. The closed-loop response θ_o/θ_i is equal to $\frac{A_p A_\theta}{1 + A_p A_\theta}$. Again, this relationship is illustrated graphically by

figure 5. The airframe-autopilot system variations which were considered in the analysis are presented in table III.

RESULTS AND DISCUSSION

Longitudinal Characteristics

Autopilot frequency response.— Experimentally determined curves showing the response of the autopilot to variable-frequency sinusoidal oscillations of different amplitudes are presented in figure 6. These curves were obtained by reading the film record to an accuracy of only ± 10 percent since a preliminary analysis indicated that the system would have a large margin of stability.

Airframe frequency response.— The calculated curves showing the response of the airframe to sinusoidal elevator deflection of variable frequency are presented in figure 7. The flight conditions shown were included so that the system response during a typical climbing condition and during high-speed flight might be determined.

Open-loop response.— The autopilot response and airframe response were combined in the manner shown under method of analysis to form the open-loop response.

These combined data are presented in the form of Nyquist diagrams. A Nyquist diagram is a polar plot of the vector frequency response. The curve plotted is the locus of the tip of the response vector. This is a compact form of presentation which allows the designer to plot all significant data in one curve on one chart. These diagrams indicate system stability if the appropriate critical point is not enclosed, while a measure of the damping may be obtained from the nearness of the locus to the critical point. This nearness may be expressed as the gain margin and phase margin. A further explanation of this form of presentation, as well as a rigorous proof of the criterion for stability, may be found in references 2, 3, and 6.

For all Nyquist diagrams presented herein, the scale of the radial coordinate is linear between values of zero and one, whereas for values greater than one a logarithmic (decibel) scale is used since a greater range of values may be plotted.

A typical open-loop response for the longitudinal mode of motion is shown in figure 8. In table IV the gain and phase margins for the various conditions considered are summarized.

Transient response.— The longitudinal transient response is determined from the closed-loop response and the response of the closed-loop system to the first half-cycle of a square wave. Such a response will very closely approximate the response to a step function. Determination

of step-function response involves the response to a Fourier integral, a procedure which is not only laborious but which presupposes an analytical expression for the response as a function of frequency. However, the response to a square wave may be determined by operating on the vector closed-loop response (which may be obtained wholly or partly by experimental means) with a Fourier series expression for a square wave and summing the result, a procedure which is relatively simple. (See reference 7.)

By this method, utilizing the closed-loop response (fig. 9) and the response to a Fourier series, the transient response to a command input signal for the cruising conditions was found. (See fig. 10.) The response is oscillatory and underdamped. Since the phase and gain margin are somewhat smaller than recommended in reference 2, page 162, a transient response of this nature may be expected.

If the ratio of gyroscope-error angle to control-surface-deflection angle is changed to a value of 7:1, the open-loop response curve shown in figure 11 is obtained. Phase and gain margin values of 58° and 74 percent will be realized under these conditions. These values are closer to the recommended margins and a much more desirable transient response, as shown in figure 12, is obtained. Table IV summarizes the gain and phase margin for various conditions considered.

On the basis of a comparison of the transient responses shown in figures 10 and 12, the ratio of gyroscope-error angle to elevator-deflection angle should be 7:1. This modification should be made by changing the linkage between the servomotor and control surface while the ratio of gyroscope-error angle to servomotor-deflection angle is held to its present value. This change will insure the retention of the present autopilot frequency-response characteristics.

Lateral Characteristics

The autopilot vector-frequency-response curves for yaw and roll are presented in figure 13. The autopilot responses for roll were determined for amplitudes of $\pm 1.5^\circ$, $\pm 3.94^\circ$, and $\pm 5.43^\circ$. For yaw the autopilot responses were determined for amplitudes of $\pm 1.3^\circ$, $\pm 4.07^\circ$, and $\pm 5.92^\circ$. The amplitudes of table oscillation in yaw differed from those in roll because an approximation, rather than a duplication, of the oscillating-table amplitude settings was expedient.

In the discussion to be presented and the longitudinal analysis previously presented, the experimental amplitude values were rounded off to 1° , 4° , and 6° . The assumption was made that the 1° , 4° , and 6° frequency responses differ from the experimental values by a negligible

amount. This assumption is justifiable since the accuracy of the experimental autopilot frequency-response curves is ± 10 percent and since, in all cases, only small changes in the frequency-response curves occur for the differences between the experimental amplitudes and the rounded figures.

The airframe lateral curves for a number of conditions are shown in figure 14. These curves represent actual physical variations which might have occurred or which might have produced advantageous or disadvantageous results in the over-all system stabilization. For instance, the angle of the principal axis of inertia might have been in error by the amount considered, or, if it had seemed advisable, $C_{n\delta_a}$ might have been changed by modifying the aileron trim position. These variations did not have an appreciable effect on the over-all system stabilization; therefore, the normal values were used for the analysis.

The phase angles of the airframe lateral frequency response were modified by subtracting 180° from the calculated values, since the sign convention employed in the calculations resulted in phase angles 180° out of phase with the true physical phase angle. The airframe lateral vector frequency responses presented herein are uncorrected. The corrected values were used in the analysis and for all lateral Nyquist plots.

In order to conduct an analysis of the lateral control system, the system must be considered as a multiple-loop servomechanism since the control system is responsive to both yaw and roll and since the airframe must roll in order to yaw. In order to use the simple Nyquist test, the stability of the inner loop must first be examined. When this loop is proved stable, the simple Nyquist criterion may be applied to the complete system. A further explanation may be found in reference 2, page 73.

Both the yaw and roll vector-frequency-response characteristics were used to determine stability. The closed-loop response for yaw was used to determine the yaw-system transient response by the method outlined for the longitudinal system.

Inner-loop stability.— A typical Nyquist diagram of the lateral system inner loop is shown in figure 15. In addition, a summary of the gain and phase margins for this loop for the various flight conditions may be found in table IV. An ample margin of stability is available for all cases except level flight at 600 miles per hour. In 600-mile-per-hour level flight at 1° amplitude, the inner loop is actually unstable. For 6° amplitude and 600-mile-per-hour level flight the margins are rather small.

System stability.— In the case where the inner loop of a multiloop servo system has been shown to be stable, the simple Nyquist test concerned with the encirclement or intersection of the critical point by the system open-loop response may be used to determine system stability. This criterion has been applied to the 250-mile-per-hour climb condition and to the 400-mile-per-hour level-flight condition, and ample stability, as indicated by large gain and phase margins, is assured in both cases. These margins are summarized in table IV and a typical system open-loop response curve is presented as figure 16.

In the case where the inner loop of a multiloop servo system is unstable, system instability is not necessarily indicated. However, by the use of a succession of Nyquist diagrams the system may be found to be stable or unstable. A more complete explanation of this case may be found in reference 2, page 73; reference 6, page 158; or reference 3, page 172. This general Nyquist criterion has been applied to the 600-mile-per-hour level-flight condition and the lateral system has been found unstable for the 1° oscillation amplitude. In addition, the inner loop has insufficient phase and gain margins for the 6° amplitude condition.

Therefore, a steady oscillation with an amplitude between 1° and 6° occurs in roll and a rather underdamped rolling motion results when a command for yaw heading change is given to the system. This instability and underdamped rolling motion constitute undesirable system operation and, for this reason, 600-mile-per-hour level flight would not be recommended.

Yaw transient response.— By use of the closed-loop response, as shown in figure 17, and a Fourier series expression for a square wave, the transient response (to a command signal input calling for a yaw heading change) was found. This response is shown in figure 18 for the 400-mile-per-hour level-flight condition.

Since the response of the system in yaw was slow, a somewhat more detailed examination of the factors influencing this response is made. The autopilot installation functions in a manner to produce zero aileron deflection when the angle of bank is equal to the instantaneous yaw heading change. This function means that in flight the aircraft will maintain an angle of bank equal to the instantaneous yaw heading change. It was known that the proportionality between the angle of bank and the instantaneous yaw heading change could be influenced by the cant angle of the gyroscope.

In order to relate this fact to the transient response, the transient response was approximated by a step-by-step method in which the bank angle was held equal to the instantaneous yaw heading change in the following expression:

$$\Delta t = \frac{V \Delta \psi}{57.3g \tan \phi}$$

where Δt is time increment in seconds and g is acceleration due to gravity. Figure 19, which shows the use of the preceding expression for a level-flight condition of 400 miles per hour, was obtained by assuming a heading of 6° . Therefore, the initial bank angle was 6° . Substituting $\phi = 6^\circ$ and $\Delta \psi = 1^\circ$ and solving for Δt results in the first point of the plot. Initial conditions for the next step, since the desired heading change is then 5° , are $\phi = 5^\circ$ and $\Delta \psi = 1^\circ$. The equation is then solved for Δt and the next step carried out in a similar manner. The time at any point is the sum of the preceding time increments. The basic approximation assumed in this method is that the turn is caused by the horizontal component of the lift vector due to the angle of bank.

The curve shown in figure 19 approximates the transient response obtained by the Fourier series method shown in figure 18. Also shown in this figure is a step-by-step approximation of the transient obtained when the bank angle is held equal to three times the instantaneous yaw heading change. The time required to reach the new yaw heading is about one-third of the time required for the present system. It seems then, that if the yaw transient response must be modified, such a change could be made by adjusting the gyroscope cant angle. The system stability could then be analyzed by the method used for the present stability determination.

CONCLUDING REMARKS

From the results of an investigation of the stability of a system composed of a subsonic canard airframe and a canted-axis gyroscope automatic pilot, the following procedure appears to be a suitable method of analysis of automatically stabilized and remotely controlled aircraft:

1. Determine the frequency-response characteristics of the airframe and control components either by calculation or by experimental means.

2. Combine the component frequency responses in the proper manner for the system considered and plot a Nyquist diagram, or series of Nyquist diagrams as required, to determine the system stability.

3. Adjust the system frequency response to the desired characteristics on the basis of the Nyquist diagrams.

4. Determine the transient response of the adjusted system to insure that the adjustments have resulted in the desired response.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., October 20, 1949

APPENDIX

AIRFRAME EQUATIONS OF MOTION AND SAMPLE CALCULATION FOR AIRFRAME

FREQUENCY-RESPONSE CHARACTERISTICS IN PITCH

The frequency-response characteristics in pitch of the airframe may be calculated by solution of the following differential equations of motion if the forward-speed effects are neglected:

$$\left. \begin{aligned} \left(\frac{I_Y}{57.3qS} D^2 - C_{mq} \frac{c}{2V} D \right) \theta - C_{m\alpha} \alpha &= C_{m\delta_e} \delta_e \\ \left(\frac{mVD}{57.3qS} - \frac{C_L \tan \gamma}{57.3} \right) \theta + \left[-\frac{mVD}{57.3qS} - \left(C_{L\alpha} - \frac{C_L \tan \gamma}{57.3} \right) \right] \alpha &= C_{L\delta_e} \delta_e \end{aligned} \right\} \quad (1)$$

where $D = \frac{d}{dt}$.

The derivative due to the lag in downwash at the wing due to the tail was omitted from the pitching-moment equation because little information is available for downwash effects on a canard airframe. All stability derivatives are in degree measure. For the level-flight condition, the solution of these equations for θ/δ_e may be obtained by setting up the determinants

$$\frac{\theta}{\delta_e} = \frac{\begin{vmatrix} C_{m\delta_e} & -C_{m\alpha} \\ C_{L\delta_e} & \left(-\frac{mVD}{57.3qS} - C_{L\alpha} \right) \end{vmatrix}}{\begin{vmatrix} \left(\frac{I_Y D^2}{57.3qS} - C_{mq} \frac{c}{2V} D \right) & -C_{m\alpha} \\ \frac{mVD}{57.3qS} & \left(-\frac{mVD}{57.3qS} - C_{L\alpha} \right) \end{vmatrix}} \quad (2)$$

Numerical values, taken from table II, are substituted in the determinants at this point to facilitate the solution. When these values are substituted, the determinants become

$$\frac{\theta}{\delta_e} = \frac{\begin{vmatrix} 0.0215 & 0.014 \\ 0.004 & (-0.0809D - 0.0825) \end{vmatrix}}{\begin{vmatrix} (0.00119D^2 + 0.000383D) & 0.014 \\ 0.0809D & (-0.0809D - 0.0825) \end{vmatrix}} \quad (3)$$

By making the substitution $D = i\omega$ and simplifying equation (3), then

$$\frac{\theta}{\delta_e} = \frac{-0.00174i\omega - 0.00184}{0.000096i\omega^3 + 0.000129\omega^2 - 0.00084i\omega}$$

This expression may be written in the vector form:

$$\frac{\theta}{\delta_e} = A + iB$$

where

$$R = \sqrt{A^2 + B^2}$$

and

$$\epsilon = \tan^{-1} \frac{B}{A}$$

The quantities R and ϵ are then plotted against ω to give the airframe vector-frequency-response curves.

The lateral equations of motion for a rudderless airframe, which may be derived from the equations given in reference 8, are as follows:

$$\left[\frac{m}{57.3qSb} (k_Z^2 \cos^2 \eta + k_X^2 \sin^2 \eta) D^2 - C_{n_r} \frac{b}{2V} D \right] \psi +$$

$$\left[\frac{m}{57.3qSb} (k_Z^2 - k_X^2) (\cos \eta \sin \eta) D^2 - C_{n_p} \frac{b}{2V} D \right] \phi - C_{n_\beta} \beta = C_{n_{\delta_a}} \delta_a$$

$$\left[\frac{m}{57.3qSb} (k_X^2 \cos^2 \eta + k_Z^2 \sin^2 \eta) D^2 - C_{l_p} \frac{b}{2V} D \right] \phi +$$

$$\left[\frac{m}{57.3qSb} (k_Z^2 - k_X^2) (\cos \eta \sin \eta) D^2 - C_{l_r} \frac{b}{2V} D \right] \psi - C_{l_\beta} \beta = C_{l_{\delta_a}} \delta_a$$

$$\left(\frac{mVD}{57.3qS} - \frac{C_L}{57.3} \tan \gamma \right) \psi - \left(\frac{C_L}{57.3} \right) \phi + \left(\frac{mVD}{57.3qS} - C_{Y\beta} \right) \beta = C_{Y_{\delta_a}} \delta_a$$

These equations are solved separately for ϕ/δ_a and ψ/δ_a by the method of determinants.

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TABLE I
CHARACTERISTICS OF THE CONTROL SYSTEM

	Longitudinal	Lateral
Pick-off dead spot, deg	1.3	2.3
Total gyroscope angular range between limit stops, deg	48	42
Static follow-up ratio (error angle to control-surface angle)	3.5:1	2.5:1
Total control-surface angular range, deg	16.2	22
Maximum rate of control-surface travel, deg per sec	11.7	19.4



TABLE II

AIRFRAME CHARACTERISTICS USED IN CALCULATING FREQUENCY RESPONSE

I_X , slug-ft ²	52
I_Z , slug-ft ²	1080
I_Y , slug-ft ²	975
k_X , ft	0.93
k_Z , ft	4.23
m , slugs	60.5
S , sq ft	21
c , ft	1.87
b , ft	11.5
V , level flight at 400 mph, ft per sec	586
C_L , level flight at 400 mph	0.255
α , level flight at 400 mph, deg	3.09
q , at 3500-ft altitude and 400 mph, lb per sq ft	364
$C_{L\alpha}$, per deg	0.083
$C_{L\delta_e}$, per deg	0.004
$C_{m\alpha}$, per deg	-0.014
$C_{m\delta_e}$, per deg	0.022
${}^1C_{mq}$, per deg	-0.24
γ , deg	0 and 3
e , deg	3.08
η , deg	$\frac{1}{2}$, 0, and $-\frac{1}{2}$
$C_{l\delta_a}$, per deg	-0.001
$C_{l\beta}$, per deg	-0.002
$C_{Y\beta}$, per deg	-0.016
$C_{n\beta}$, per deg	0.004
${}^1C_{np}$, per deg	-0.00007
${}^1C_{lp}$, per deg	-0.0112
${}^1C_{nr}$, per deg	-0.0122
$C_{n\delta_a}$, per deg	0.0002 and -0.0002
$C_{Y\delta_a}$	0
${}^1C_{lr}$, per deg	0.0003

¹These values were estimated, whereas other stability derivatives were obtained from unpublished wind-tunnel data.

TABLE III
AIRFRAME-AUTOPILOT SYSTEM VARIATIONS INVESTIGATED

Parameter	Range of variation	Reason for consideration
Flight condition	Climb at 250 miles per hour Level flight at 400 miles per hour Level flight at 600 miles per hour	Influence of possible flight conditions on system stability. (Values of aerodynamic derivatives assumed to hold for all conditions considered)
$C_{\eta a}$	0.0002 and -0.0002	Possible improvement in system operation
η	$0^\circ, -\frac{1^\circ}{2}, \frac{1^\circ}{2}$	Effect of possible error in calculated value due to method of calculation
Autopilot input amplitude	$\pm 1^\circ, \pm 4^\circ, \pm 6^\circ$	Nonlinearity of response with amplitude variations
Pitch follow-up ratio (gyroscope-error angle: elevator angle)	3.5:1, 7:1	Improvement of stability and transient response



TABLE IV

SUMMARY OF GAIN AND PHASE MARGINS OBTAINED FROM THE OPEN-LOOP RESPONSE

Flight condition	Autopilot input amplitude (deg)	Gain margin (percent)	Phase margin (deg)
Longitudinal results; follow-up ratio ^a 3.5:1			
Climb at 250 miles per hour	$\begin{cases} \pm 1 \\ \pm 4 \\ \pm 6 \end{cases}$	$\begin{matrix} 100 \\ 78 \\ 77 \end{matrix}$	$\begin{matrix} 26 \\ 26 \\ 36 \end{matrix}$
Level flight at 400 miles per hour	$\begin{cases} \pm 1 \\ \pm 4 \\ \pm 6 \end{cases}$	$\begin{matrix} 100 \\ 50 \\ 58 \end{matrix}$	$\begin{matrix} 20 \\ 19 \\ 25 \end{matrix}$
Level flight at 600 miles per hour	$\begin{cases} \pm 1 \\ \pm 4 \\ \pm 6 \end{cases}$	$\begin{matrix} 100 \\ 20 \\ 30 \end{matrix}$	$\begin{matrix} 21 \\ 10 \\ 16 \end{matrix}$
Longitudinal results; follow-up ratio 7:1			
Climb at 250 miles per hour	$\begin{cases} \pm 1 \\ \pm 6 \end{cases}$	$\begin{matrix} 90 \\ 90 \end{matrix}$	$\begin{matrix} 93 \\ 93 \end{matrix}$
Level flight at 400 miles per hour	$\begin{cases} \pm 1 \\ \pm 4 \\ \pm 6 \end{cases}$	$\begin{matrix} 100 \\ 74 \\ 80 \end{matrix}$	$\begin{matrix} 60 \\ 60 \\ 68 \end{matrix}$
Level flight at 600 miles per hour	$\begin{cases} \pm 1 \\ \pm 6 \end{cases}$	$\begin{matrix} 100 \\ 65 \end{matrix}$	$\begin{matrix} 45 \\ 55 \end{matrix}$
Lateral results; inner-loop response			
Climb at 250 miles per hour	$\begin{cases} \pm 1 \\ \pm 6 \end{cases}$	$\begin{matrix} 55 \\ 70 \end{matrix}$	$\begin{matrix} 70 \\ 75 \end{matrix}$
Level flight at 400 miles per hour	$\begin{cases} \pm 1 \\ \pm 6 \end{cases}$	$\begin{matrix} 26 \\ 40 \end{matrix}$	$\begin{matrix} 35 \\ 39 \end{matrix}$
Level flight at 600 miles per hour	$\begin{cases} \pm 1 \\ \pm 6 \end{cases}$	$\begin{matrix} {}^b-17 \\ 9 \end{matrix}$	$\begin{matrix} -30 \\ 7 \end{matrix}$
Lateral results; system open-loop response			
Climb at 250 miles per hour	$\begin{cases} \pm 1 \\ \pm 6 \end{cases}$	$\begin{matrix} 85 \\ 88 \end{matrix}$	$\begin{matrix} 85 \\ 85 \end{matrix}$
Level flight at 400 miles per hour	$\begin{cases} \pm 1 \\ \pm 6 \end{cases}$	$\begin{matrix} 90 \\ 89 \end{matrix}$	$\begin{matrix} 88 \\ 85 \end{matrix}$
Level flight at 600 miles per hour	$\begin{cases} \pm 1 \\ \pm 6 \end{cases}$	$\begin{matrix} {}^b- \\ 91 \end{matrix}$	$\begin{matrix} - \\ 88 \end{matrix}$

^a $\frac{\text{Gyroscope error angle}}{\text{Control-surface deflection}} = \text{Follow-up ratio.}$

^b Unstable condition.



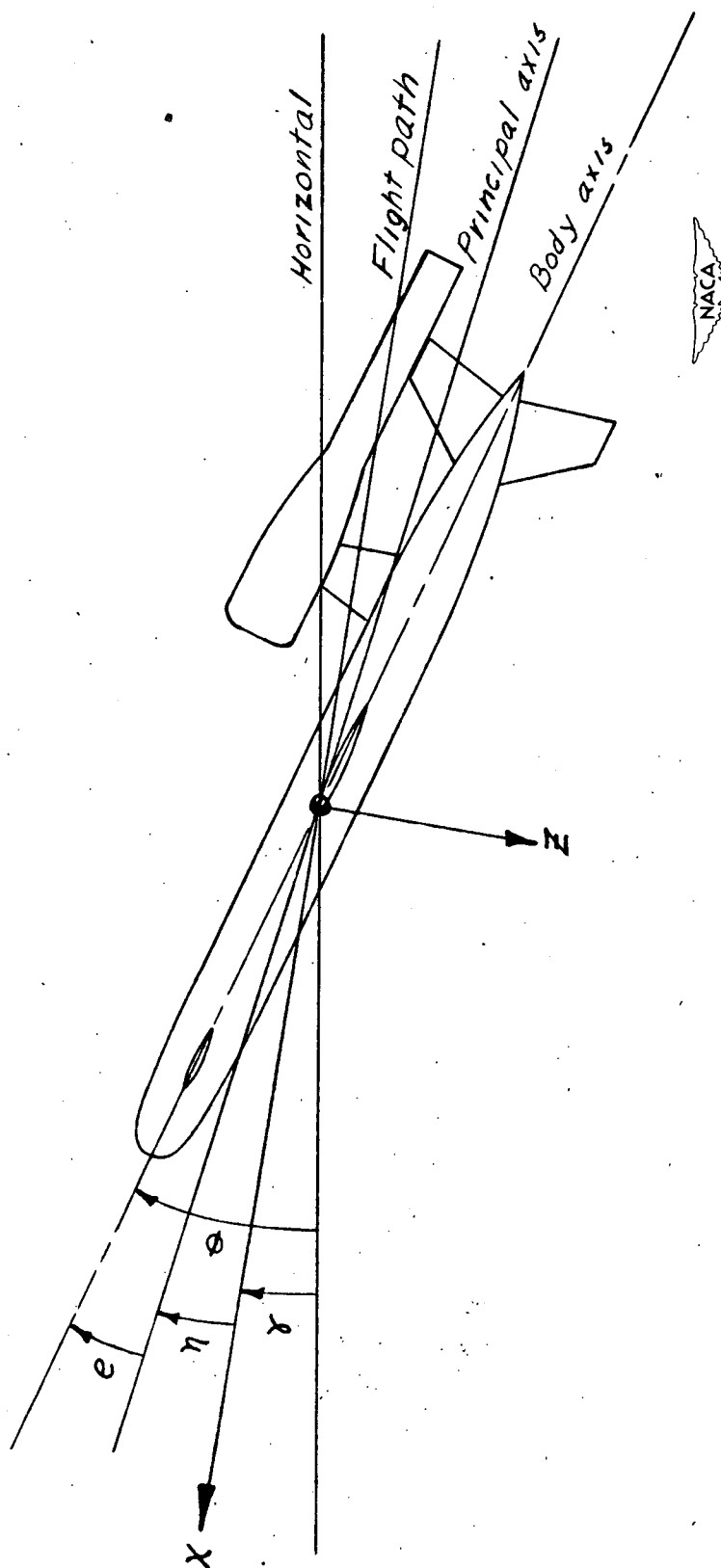


Figure 1.- Longitudinal airframe axes.

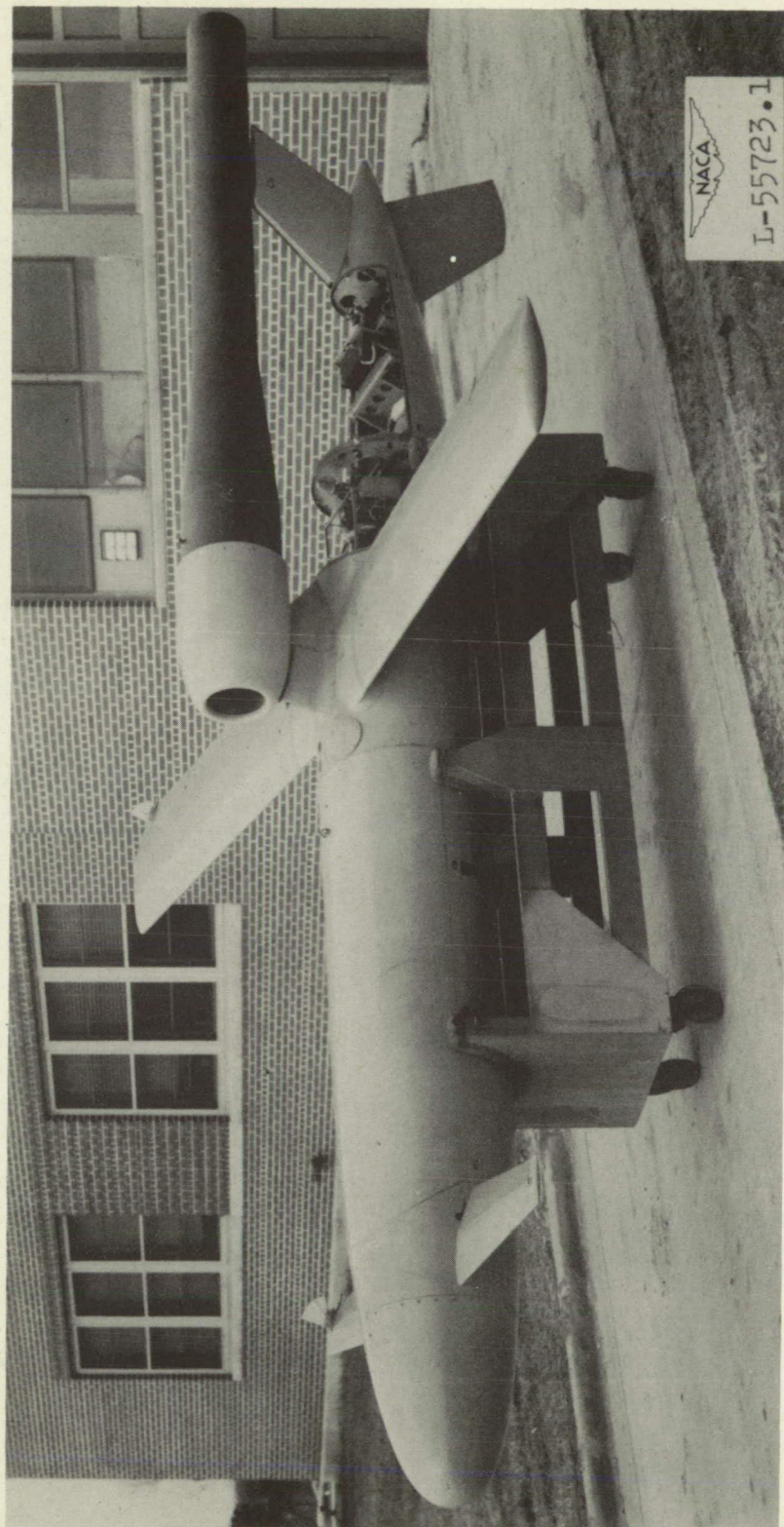


Figure 2.- Canard airframe with rear hatches removed.

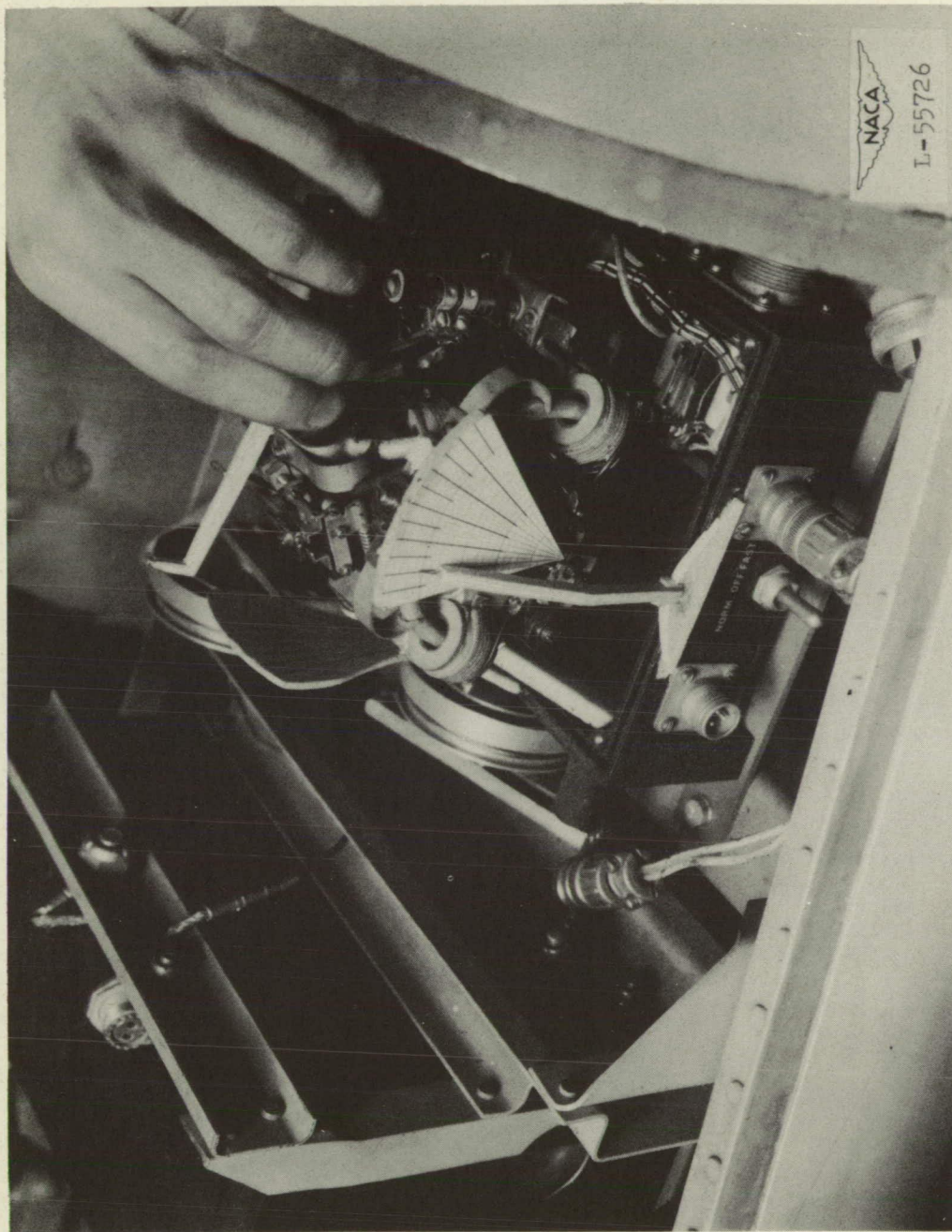


Figure 3.- Autopilot installed in airframe with protractors used for measuring autopilot installation characteristics.

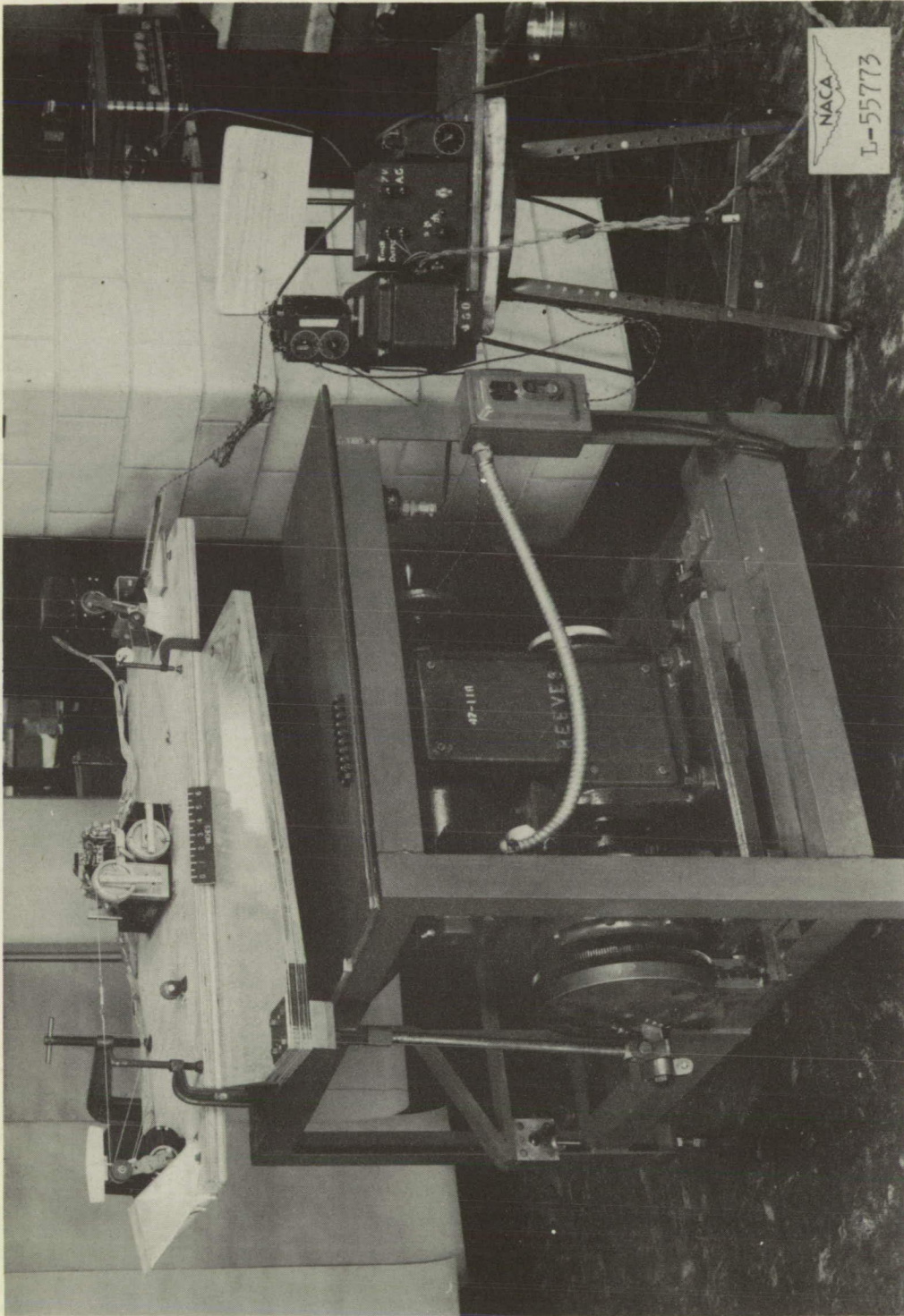


Figure 4.- Autopilot test setup showing oscillating table and recording instruments.

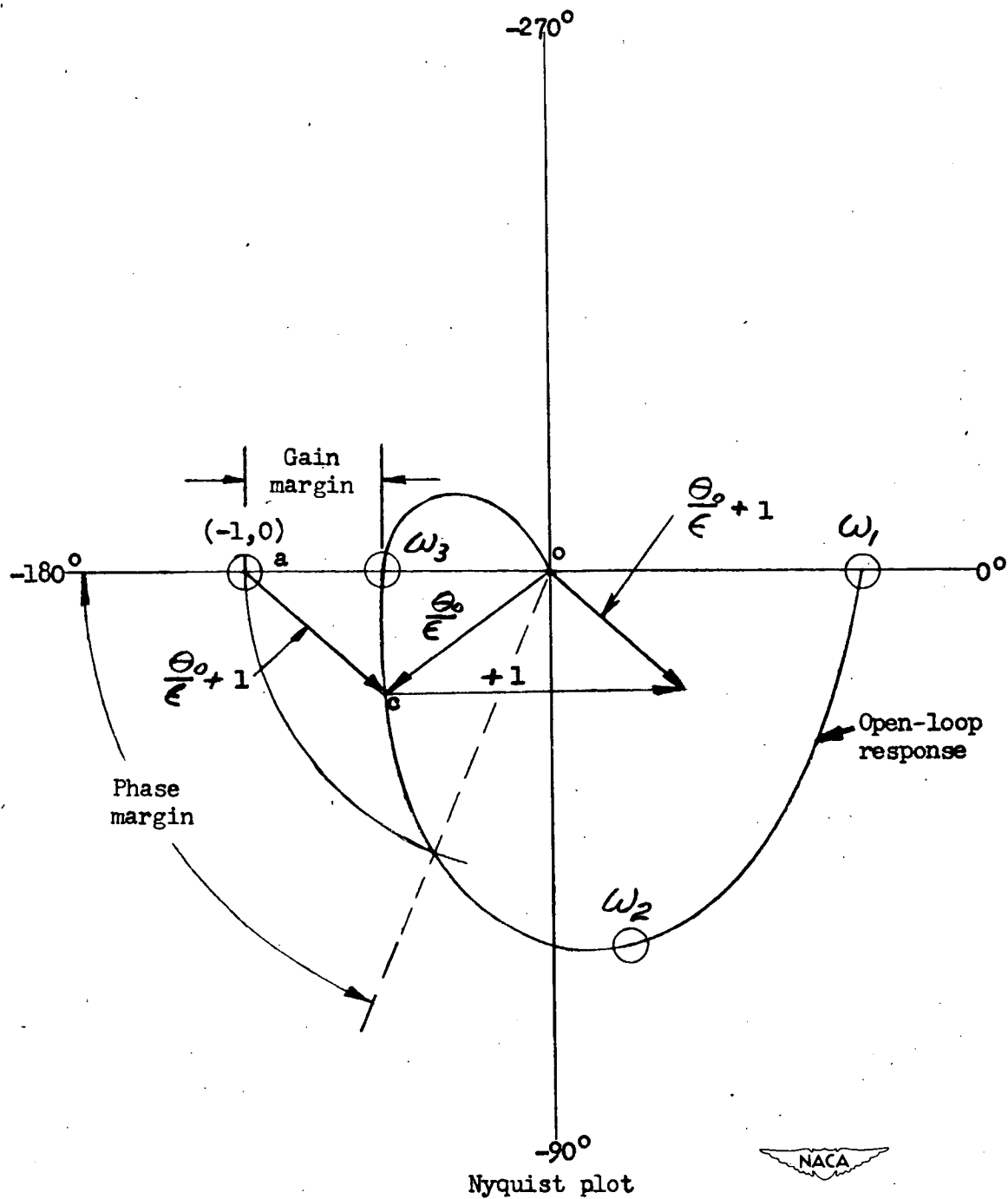


Figure 5.- Graphical relationship between open-loop and closed-loop frequency-response vectors. Closed-loop response = $\frac{OC}{AC}$.

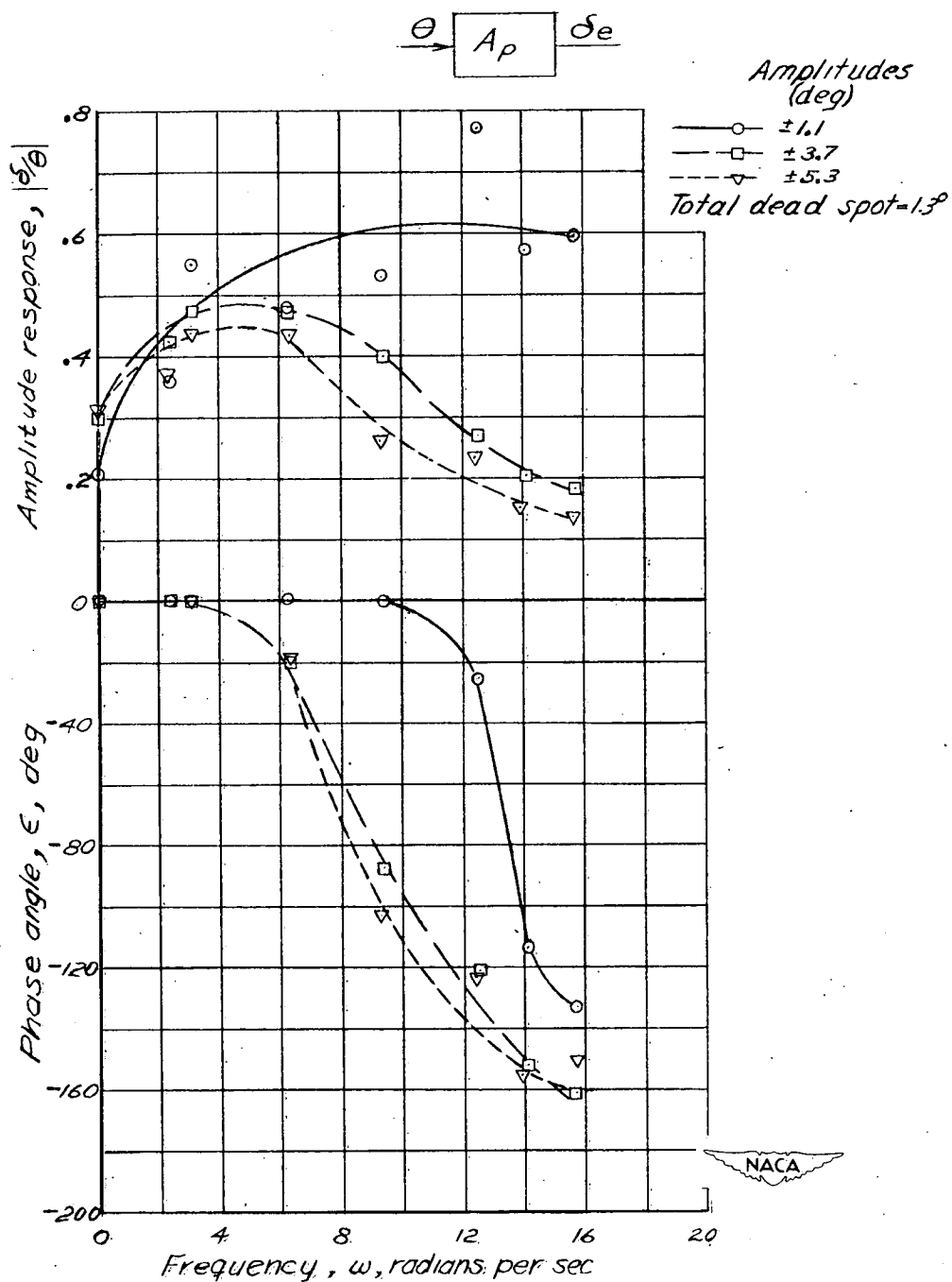


Figure 6.- Autopilot longitudinal vector-frequency-response characteristics.

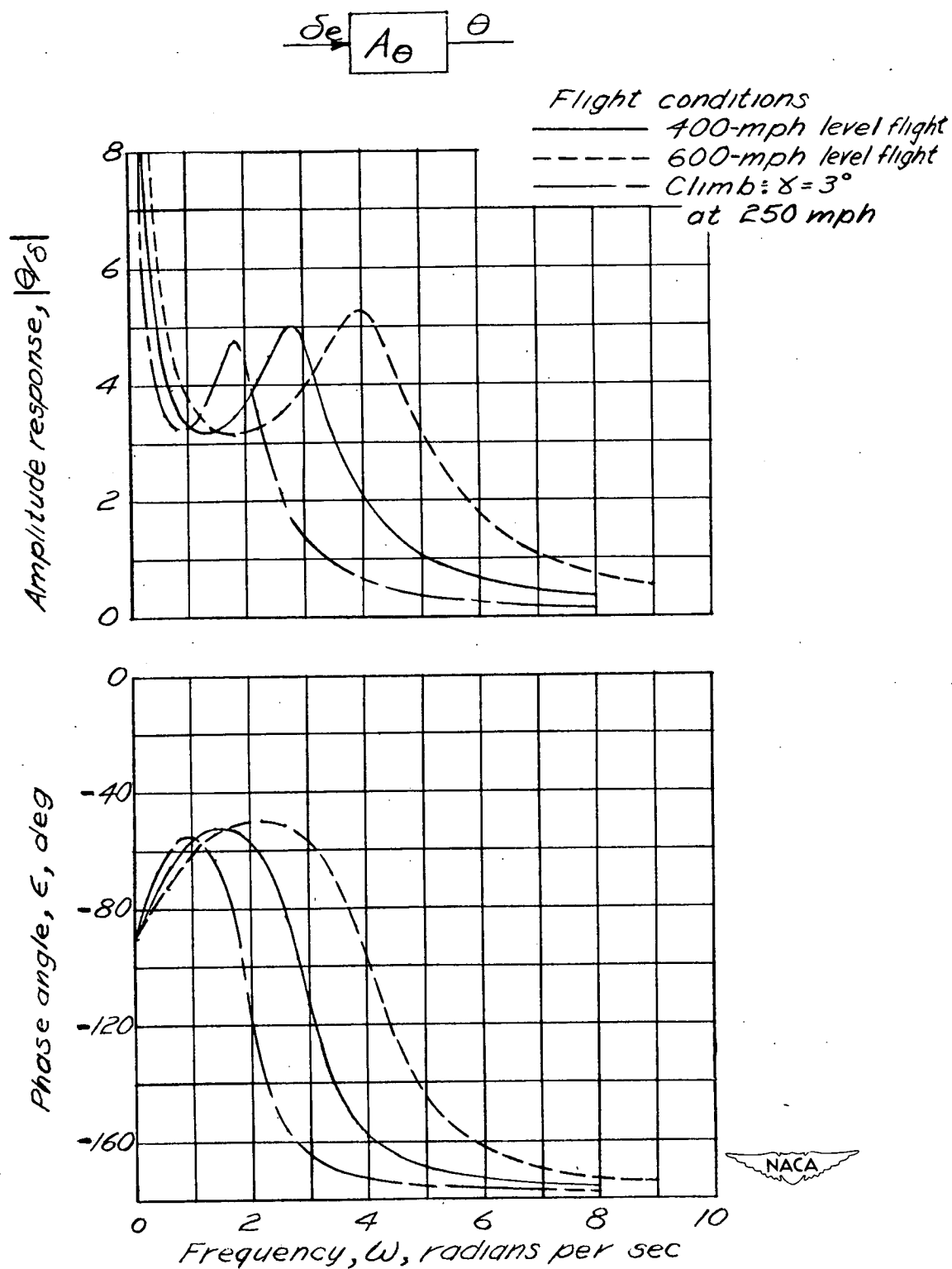


Figure 7.- Airframe longitudinal vector-frequency-response characteristics.

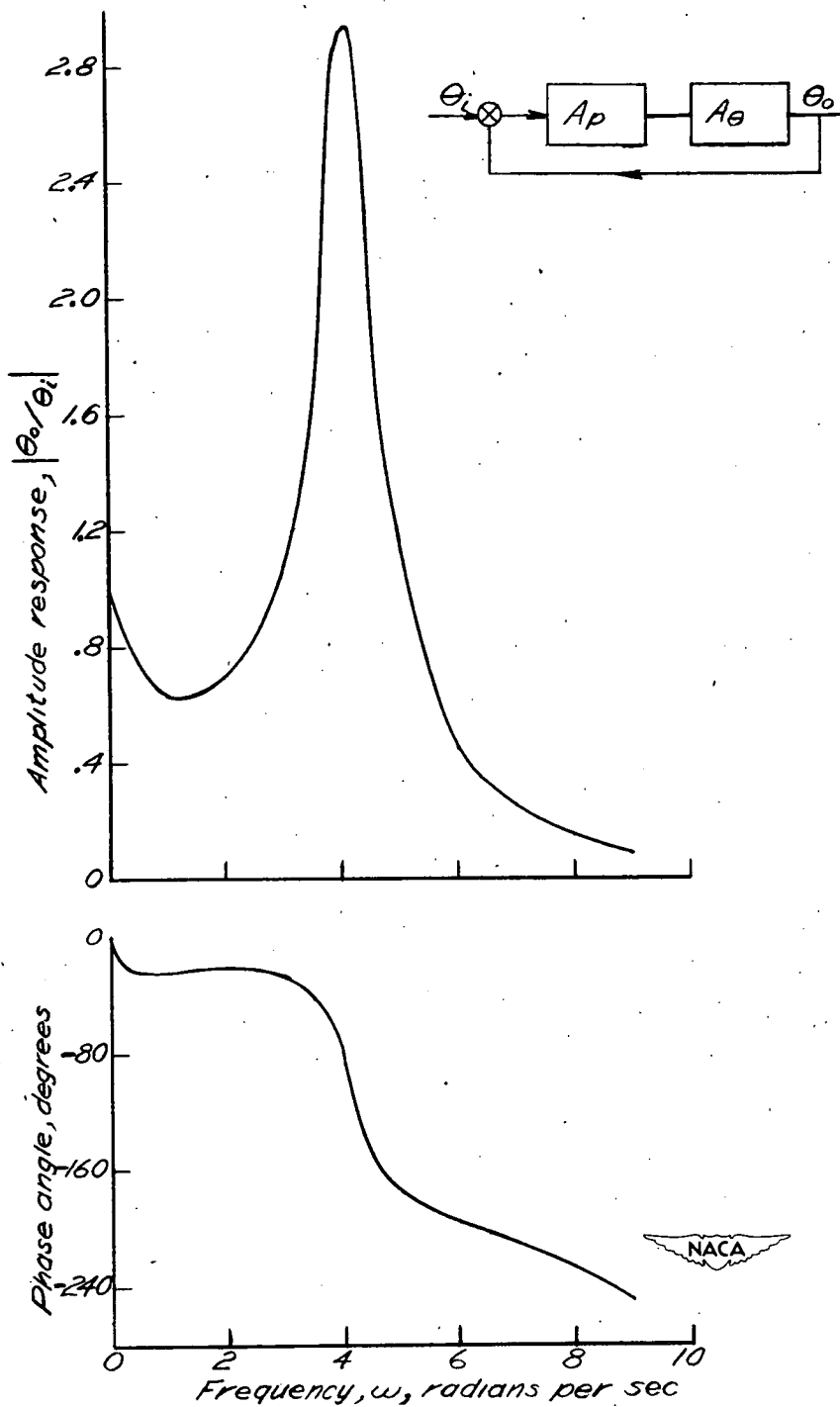


Figure 9.- Longitudinal closed-loop vector frequency response for $\pm 4^\circ$ autopilot input amplitude and 400-mile-per-hour level flight. Ratio of gyroscope-error angle to elevator angle equals 3.5 to 1.

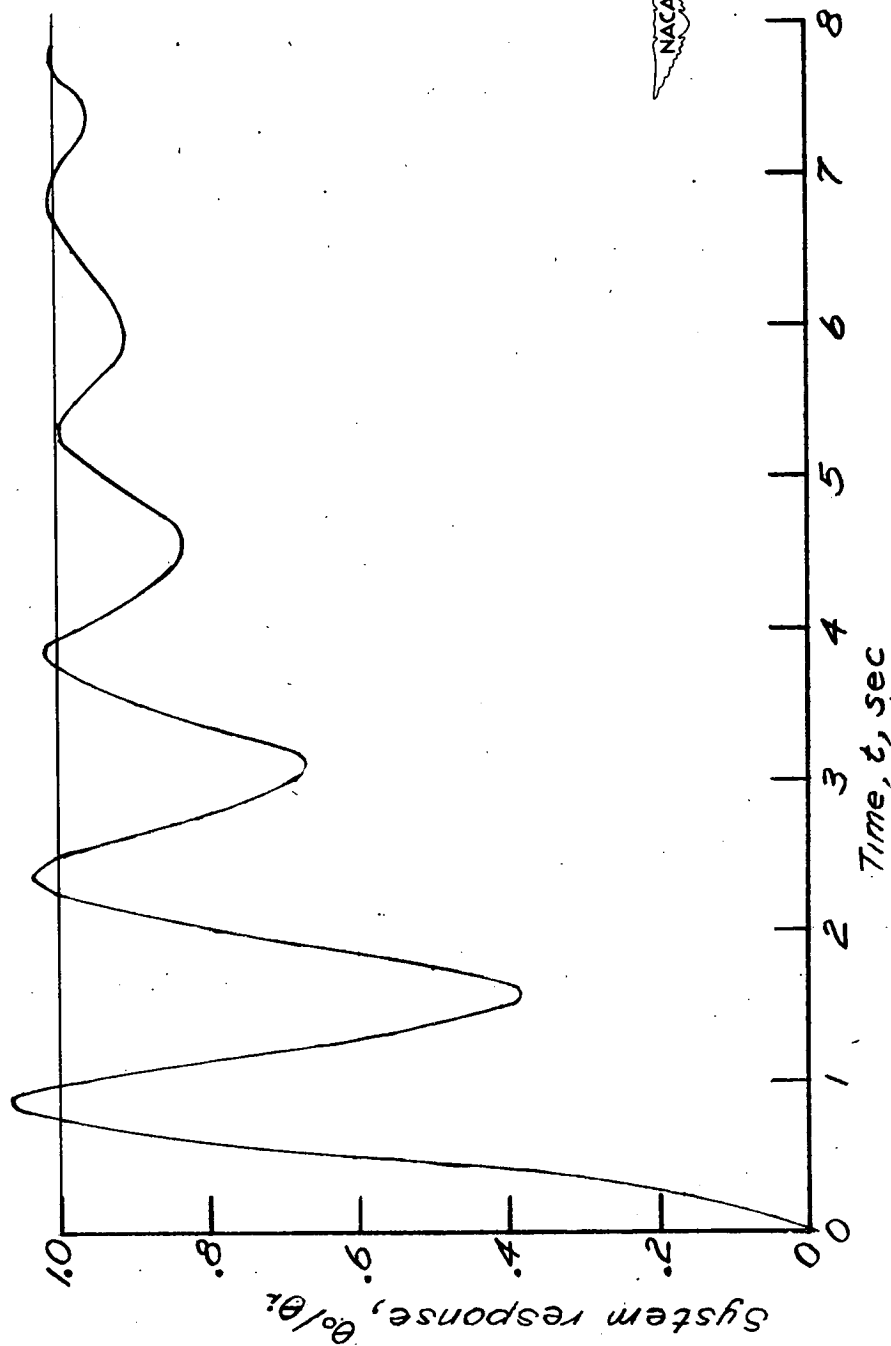
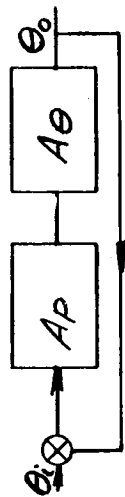


Figure 10.- Longitudinal transient response of the airframe and autopilot system to a command signal calling for a sudden change of attitude of 4° during 400-mile-per-hour level flight. Ratio of gyroscope-error angle to elevator angle equals 3.5 to 1.

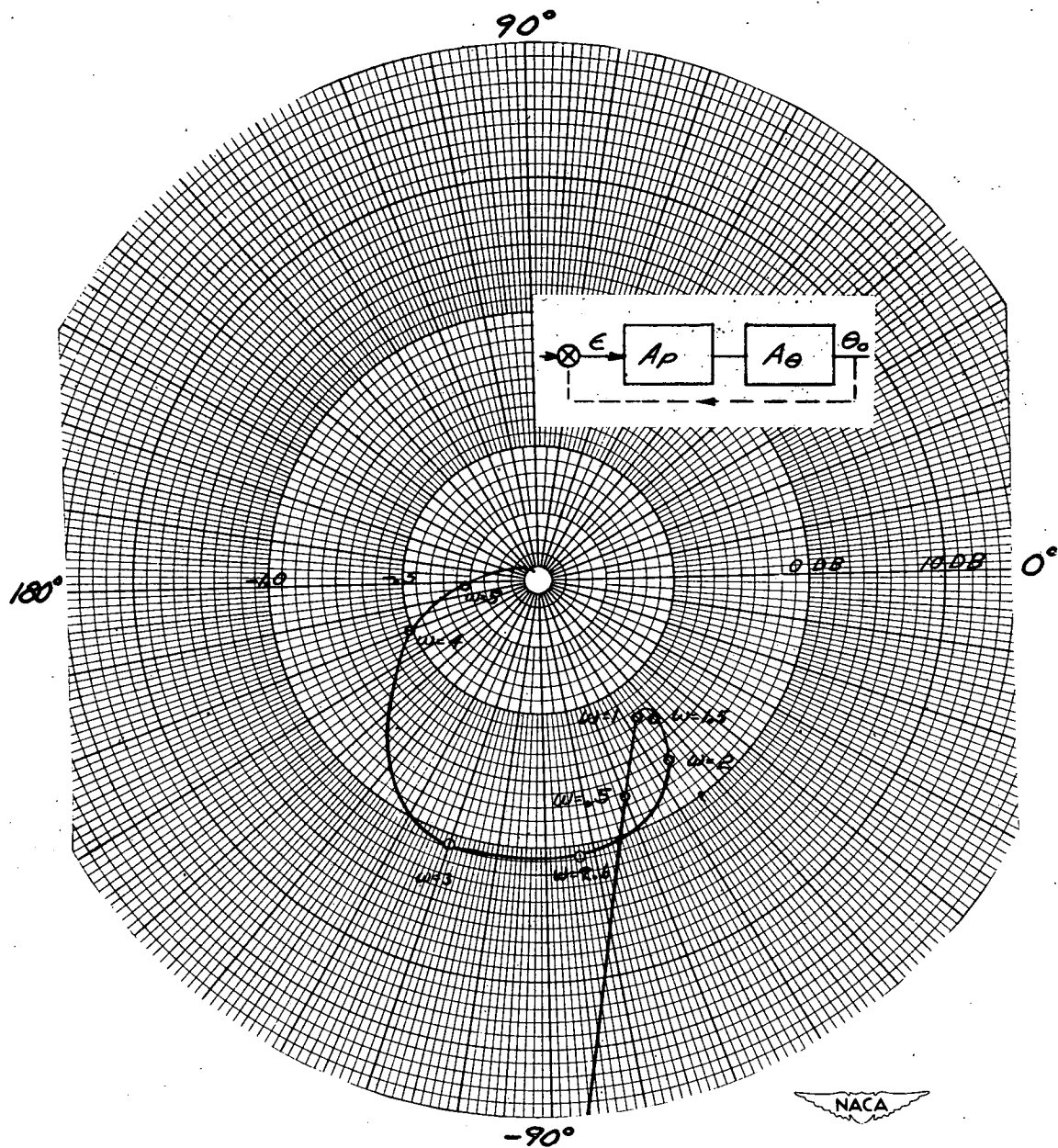


Figure 11.- Nyquist diagram showing the longitudinal open-loop frequency response for $\pm 4^\circ$ autopilot input amplitude and 400-mile-per-hour level flight. Ratio of gyroscope-error angle to elevator angle equals 7 to 1.

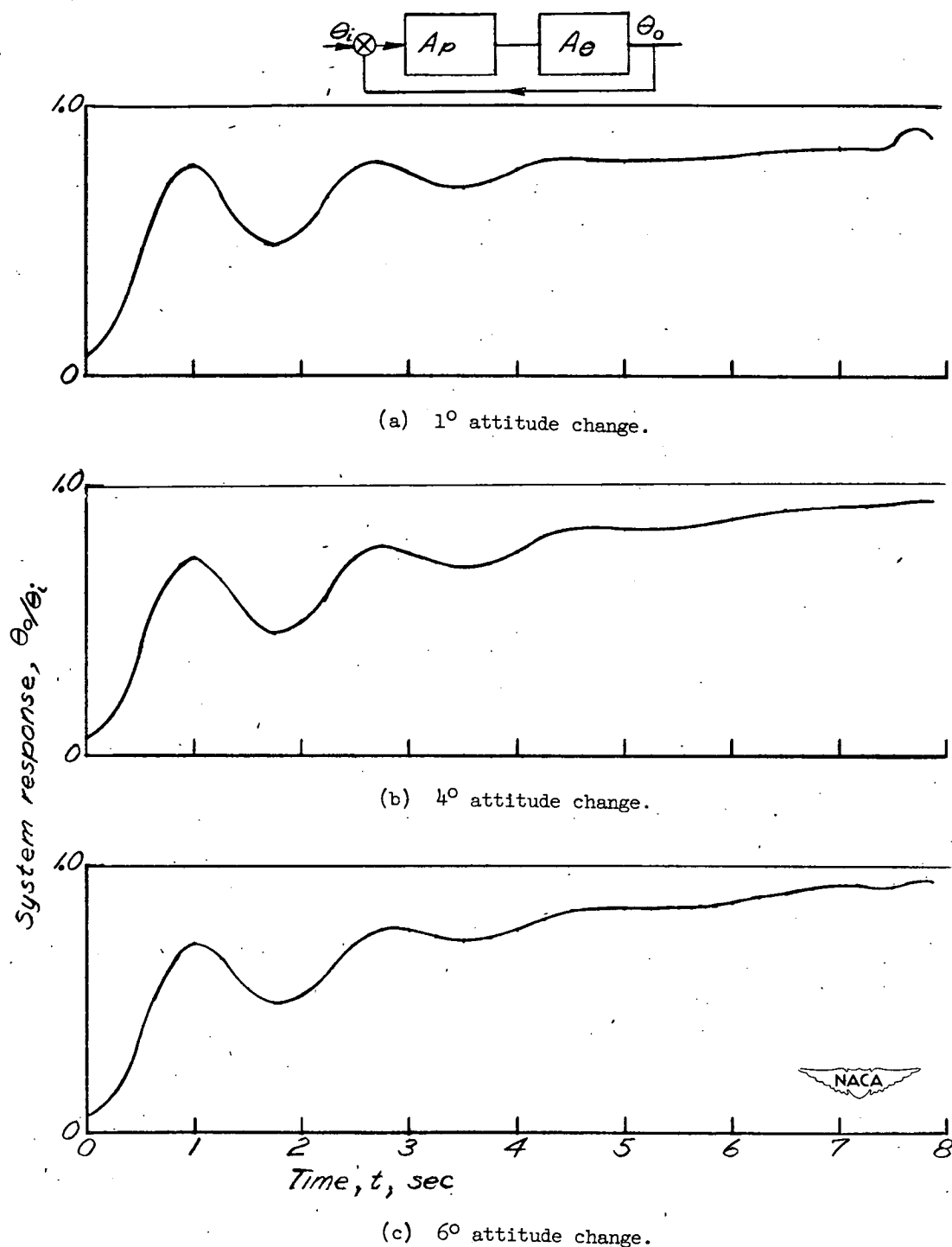
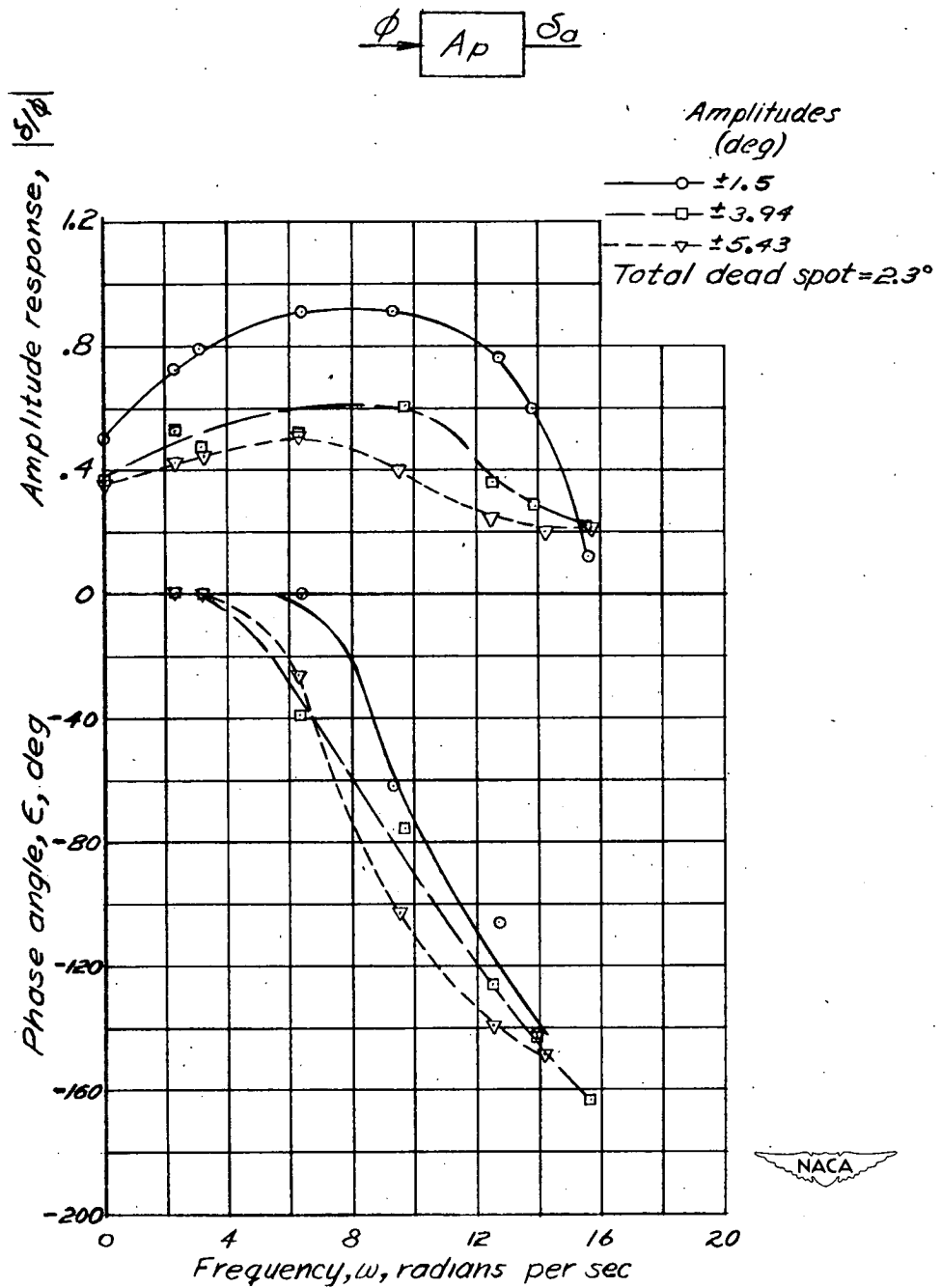
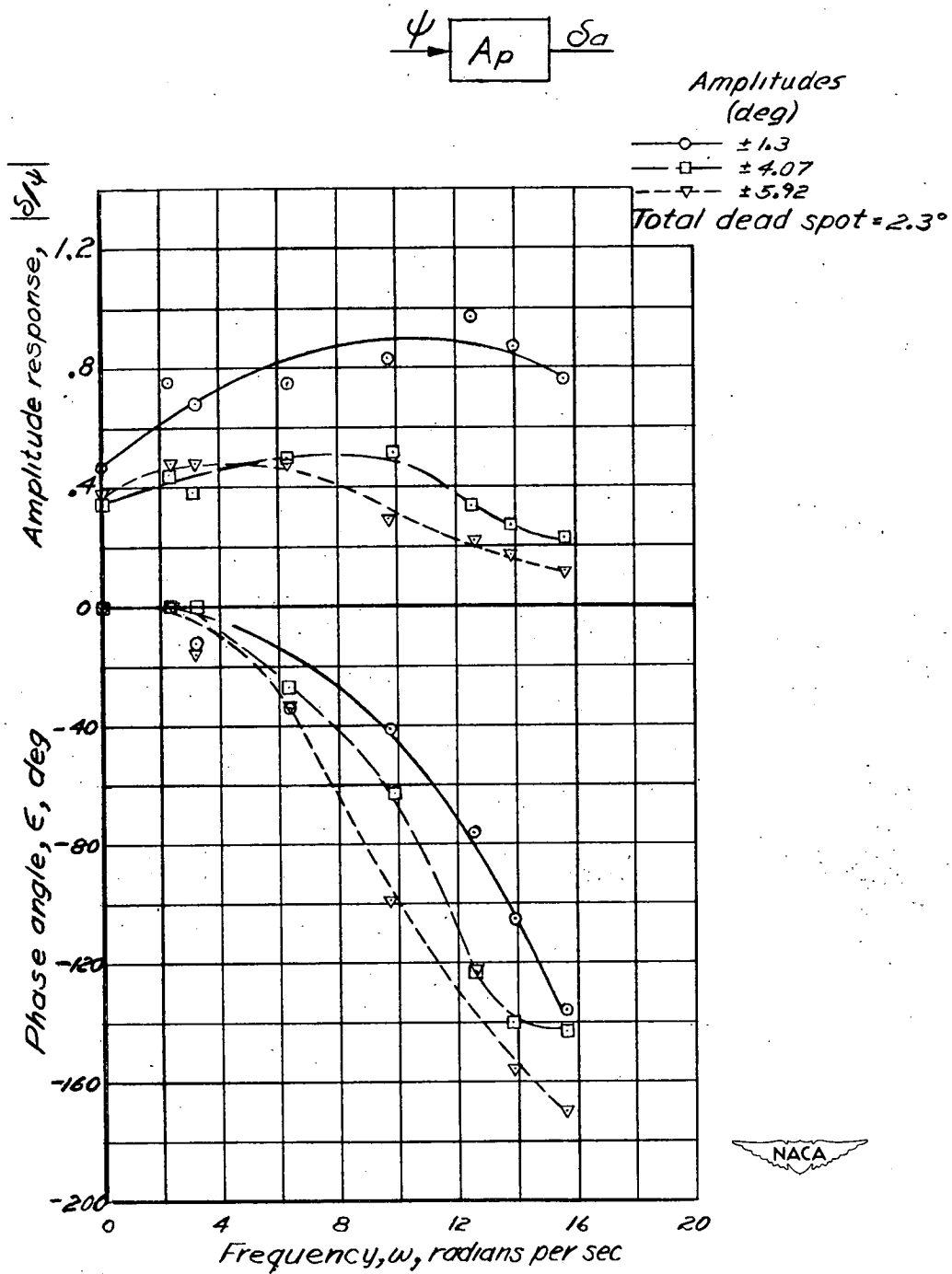


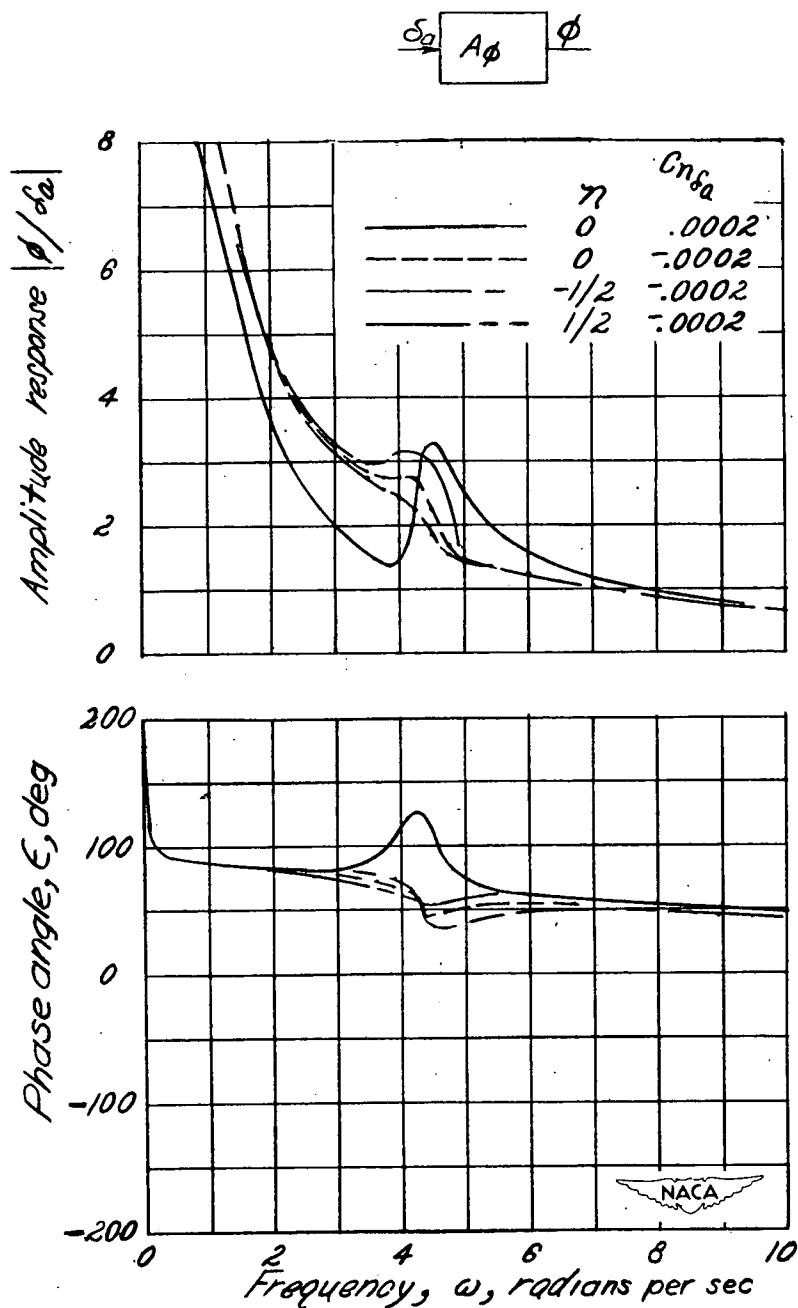
Figure 12.- Longitudinal transient response of the system to a command signal calling for a sudden change in attitude during 400-mile-per-hour level flight. Ratio of gyroscope-error angle to control-surface angle equals 7 to 1.



(a) Roll response.

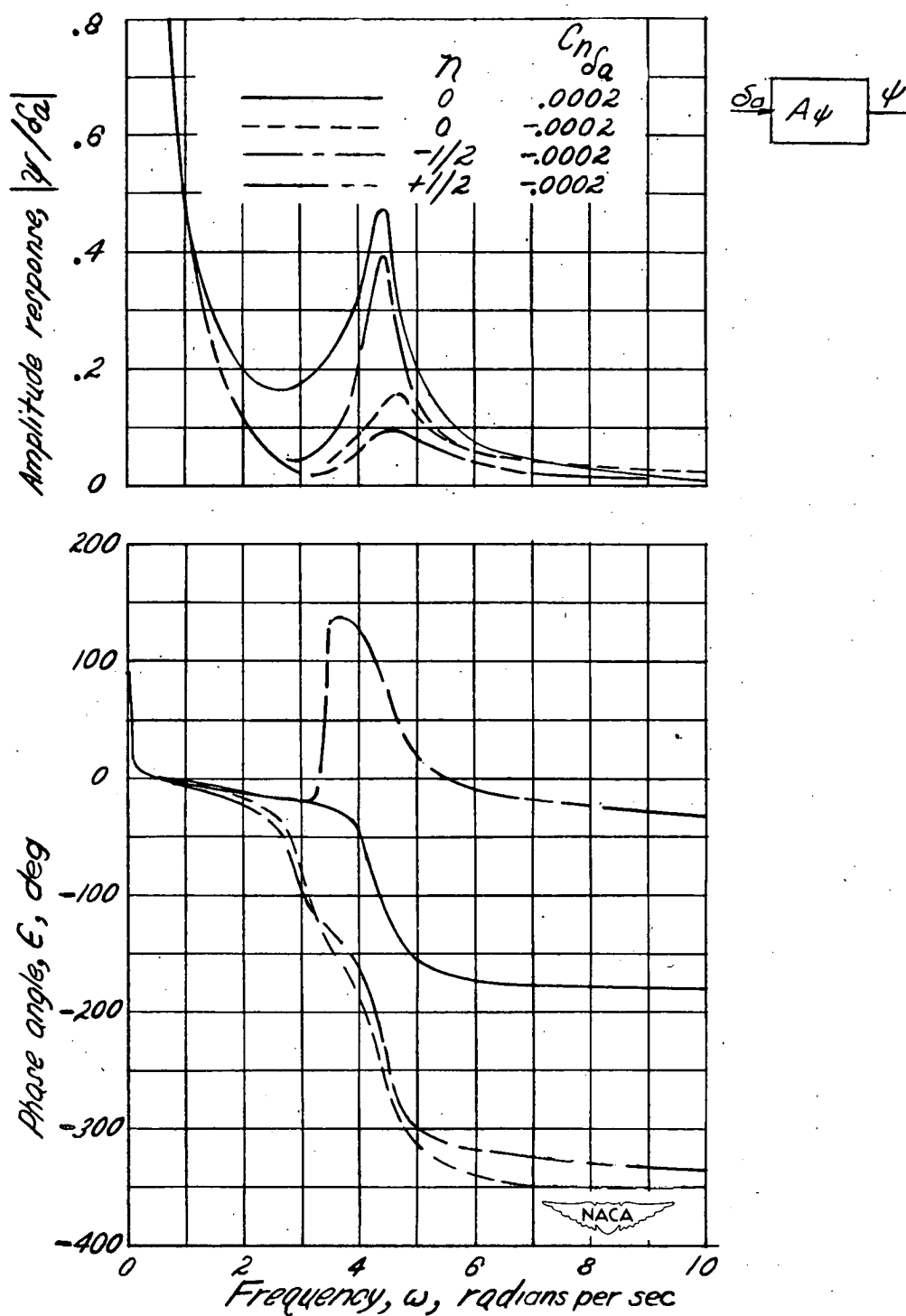
Figure 13.- Autopilot lateral vector-frequency-response characteristics.





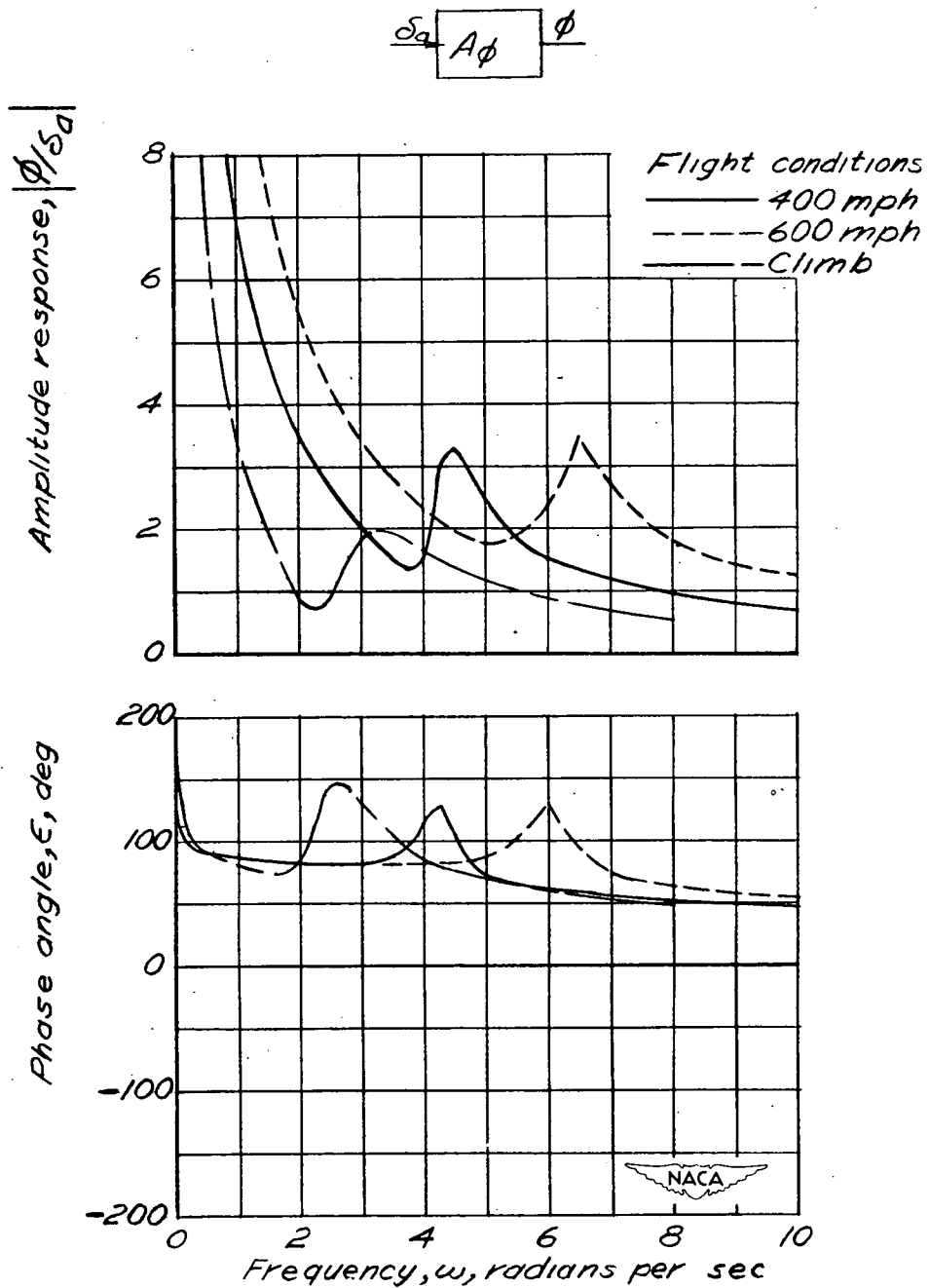
(a) Roll response for 400-mile-per-hour level flight.

Figure 14.- Lateral airframe frequency-response characteristics showing the effect of parameter variations.



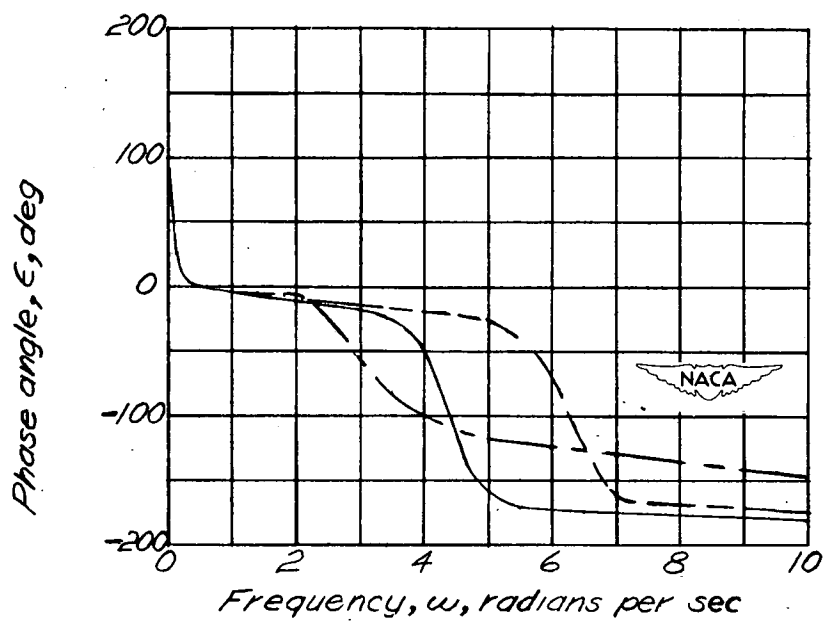
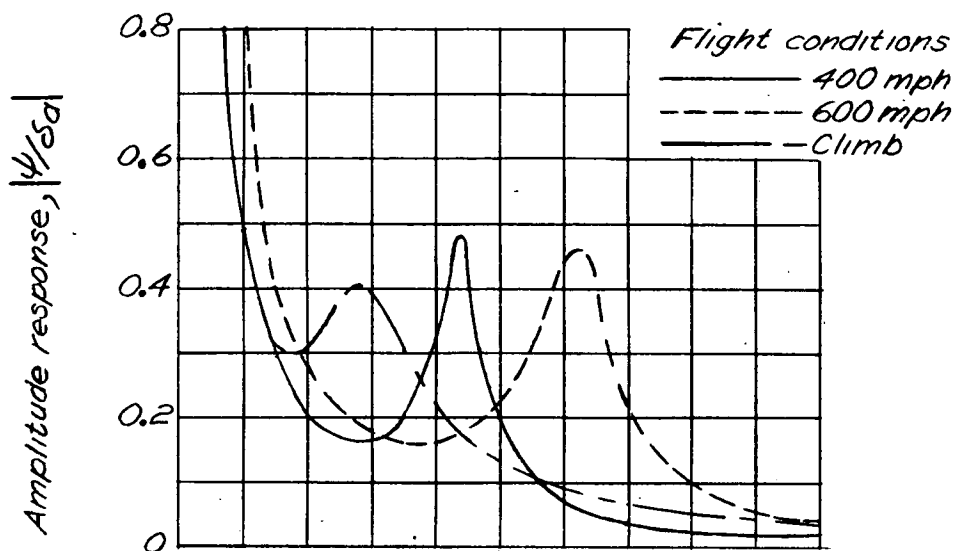
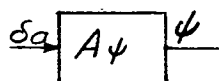
(b) Yaw response for 400-mile-per-hour level flight.

Figure 14.- Continued.



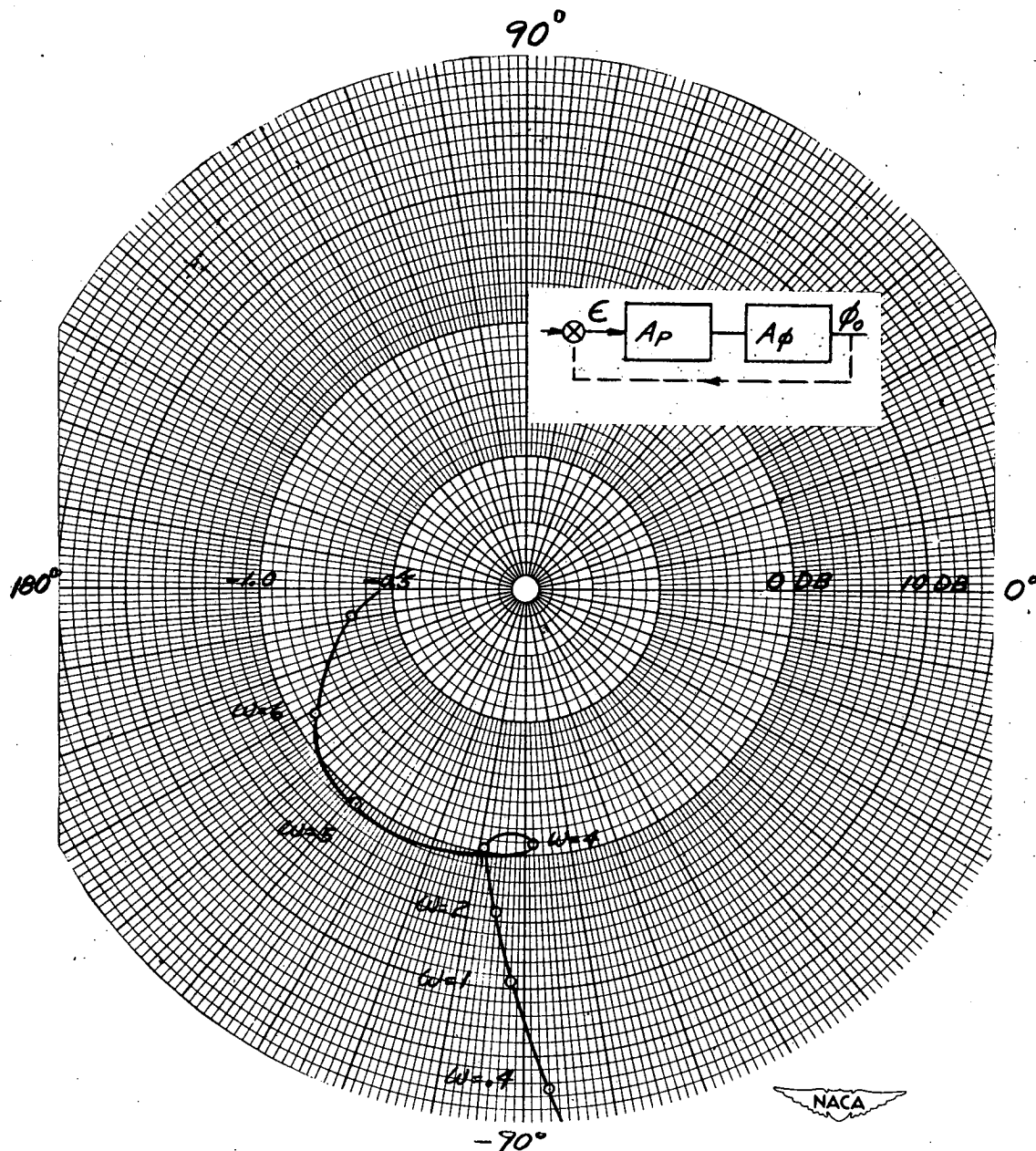
(c) Roll response. $\eta = 0^\circ$, $C_{n\delta_a} = 0.0002$.

Figure 14.- Continued.



(d) Yaw response. $\eta = 0^\circ$, $C_{n\delta_a} = 0.0002$.

Figure 14.- Concluded.



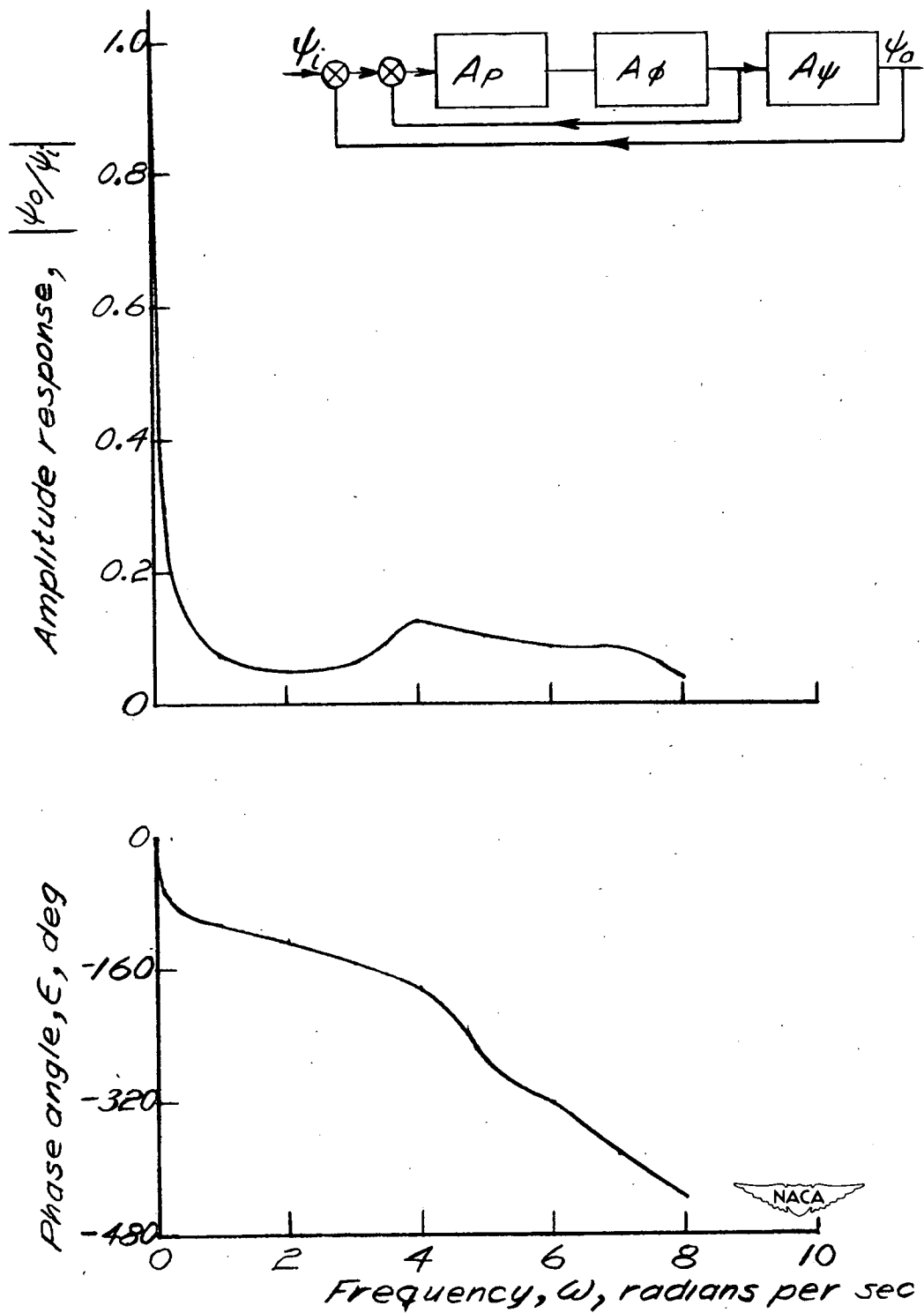


Figure 17.- Yaw closed-loop vector frequency response. Autopilot amplitude = $\pm 6^\circ$; 400-mile-per-hour level flight; $\eta = 0^\circ$; $Cn_{\delta_a} = 0.0002$.

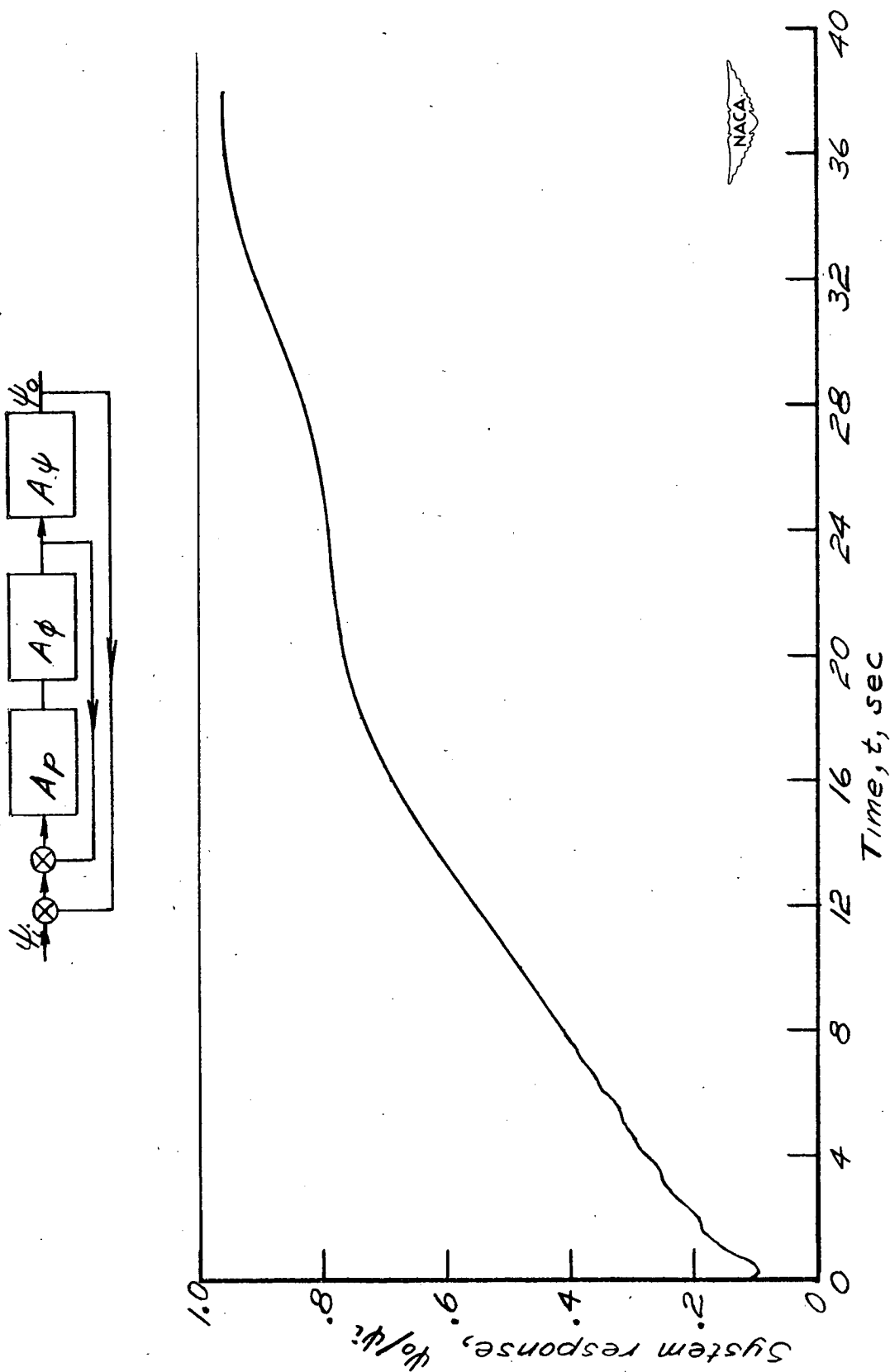


Figure 18.- Yaw transient response of the system to a command signal calling for a sudden change in heading of 6°. 400-mile-per-hour level flight; $\eta = 0^\circ$; $C_{n\delta_a} = 0.0002$.

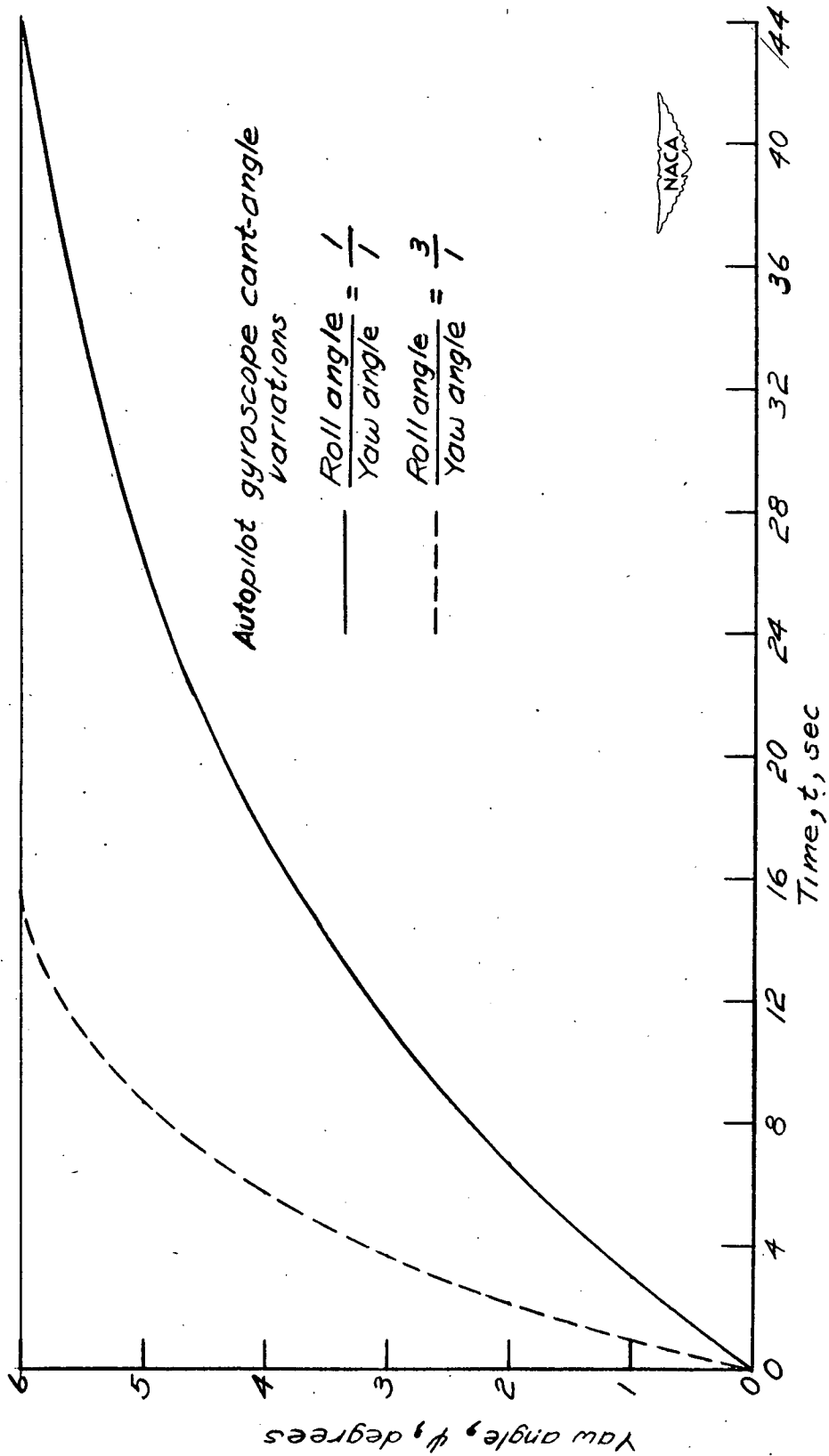


Figure 19.- Approximate yaw transient response due to a command signal (obtained by step-by-step calculation) and showing the effect of gyroscope cant-angle variations.